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The period function of generalized Loud's centers

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Abstract. In this paper a three parameter family of planar differential systems with homogeneous nonlinearities of arbitrary odd degree is studied. This family is an extension to higher degree of the Loud's systems. The origin is a nondegenerate center for all values of the parameter and we are interested in the qualitative properties of its period function. We study the bifurcation diagram of this function focusing our attention on the bifurcations occurring at the polycycle that bounds the period annulus of the center. Moreover we determine some regions in the parameter space for which the corresponding period function is monotonous or it has at least one critical period, giving also its character (maximum or minimum). Finally we propose a complete conjectural bifurcation diagram of the period function of these generalized Loud's centers.

1 Introduction and statement of the results

The present paper deals with planar polynomial ordinary differential systems and we study the qualitative properties of the period function of centers. Recall that a critical point p of a planar differential system is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest neighbourhood with this property is called the *period annulus* of the center and in what follows it will be denoted by $\mathscr{P}$. The *period function* assigns to each periodic orbit in $\mathscr{P}$ its period. If the period function is constant then the center is called *isochronous*. The study of the period function is a nontrivial problem and questions related to its behaviour have been extensively studied. Let us quote, for instance, the problems of isochronicity (see [6, 7, 25]), monotonicity (see [3, 23, 27]) or bifurcation of critical periods (see [4, 12, 13]). Aside from the intrinsic interest of these problems, the study of the period function is also important in the analysis of nonlinear boundary value problems and in perturbation theory. Indeed, for instance, under the condition of non-criticality of the period function, zeros of appropriate Melnikov functions guarantee the persistence of subharmonic periodic orbits of a Hamiltonian system after a small periodic non-autonomous perturbation (see [8, 15]). Most of the work on planar polynomial differential systems, including the present paper, is related to questions surrounding the well known Hilbert's 16th problem (see [9, 11, 22, 26] and references therein) and its various weakened versions.

Chicone [5] has conjectured that if a quadratic system has a center with a period function which is not monotonic then, by an affine transformation and a constant rescaling of time, it can be brought to the Loud normal form

$$\begin{cases} \dot{x} = -y + Bxy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases}$$

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and that the period function of these centers has at most two critical periods. The authors together with P. Mardešić have devoted a series of papers (see [17, 18, 19, 20, 23]) addressed to prove the second part of Chicone's conjecture. In the present paper we widen the scope of the study to a higher degree family maintaining the reversibility, which in particular guarantees the origin to be a center, and the homogeneity of the nonlinearities. More precisely, we study the period function of the center at the origin of the planar differential system

$$(\mathscr{L}_{n,\mu}) \qquad \begin{cases} \dot{x} = -y + Bx^{n-1}y, \\ \dot{y} = x + Dx^n + Fx^{n-2}y^2, \end{cases} \tag{1}$$

with $\mu := (B, F, D) \in \mathbb{R}^3$ and an *odd* natural number $n \geqslant 3$. To the best of our knowledge this family of differential systems was first considered in [2], where the authors studied the isochronicity problem for an arbitrary natural number n. In a subsequent paper (see [24]), by analyzing the ideal generated by the *period constants*, it is proved that $(\mathscr{L}_{n,\mu})$ has at most one (respectively, two) critical periods near the origin for n odd (respectively, even). Expecting this local bound to be global, in this paper we tackle the case n odd. This case has the additional advantage that the phase portrait of $(\mathscr{L}_{n,\mu})$ is symmetric with respect to both coordinate axes and, as we will see, this simplifies the computations that we must carry through.

Compactifying $\mathbb{R}^2$ to the Poincaré disc, the boundary of the period annulus $\mathscr{P}$ of the center has two connected components, the center itself and a polycycle. We call them, respectively, the *inner* and *outer boundary* of the period annulus. Since period function is defined on the set of periodic orbits in $\mathscr{P}$, usually the first step is to parametrize this set, let us say $\{\gamma_s\}_{s\in(0,1)}$, so that one can study the qualitative properties of the period function by means of the map $s \longmapsto$ period of γ_s, which is smooth on $(0, 1)$. The *critical periods* are the critical points of this function and its number, character (maximum or minimum) and distribution do not depend on the particular parametrization of the set of periodic orbits used. For each fixed n, the differential system under consideration depends on a three-dimensional parameter μ, and our aim is to decompose $\mathbb{R}^3 = \cup V_i$ so that if μ_1 and μ_2 belong to the same set V_i, then the corresponding period functions are qualitatively the same (i.e., their critical periods are equal in number, character and distribution.) A parameter $\mu_0 \in \mathbb{R}^3$ is a *regular value* if it belongs to the interior of some V_i, otherwise it is a *bifurcation value*. The set of bifurcation values is $\mathscr{B} := \cup \partial V_i$ and, roughly speaking, it consists of those parameters $\mu_0 \in \mathbb{R}^3$ for which some critical period emerges or disappears as μ tends to μ_0. There are three different situations to consider:

(a) Bifurcations of critical periods from the inner boundary (i.e., the center).

(b) Bifurcations of critical periods from the interior of the period annulus.

(c) Bifurcations of critical periods from the outer boundary (i.e., the polycycle).

We refer the reader to [18] for the definition of these notions.

One can readily verify that the change of coordinates $\{u = \lambda x, v = \lambda y\}$ brings the differential system $(\mathscr{L}_{n,\mu})$ into $(\mathscr{L}_{n,\hat{\mu}})$ with $\hat{\mu} = \frac{1}{\lambda^{n-1}}\,\mu$. Hence, since n is odd, the parameter space has the following *radial property.* For each $\mu_0 \in \mathbb{R}^3$ the period function of the differential system $(\mathscr{L}_{n,t\mu_0})$ is, up to a reparameterization of the set of periodic orbits, the same for any $t > 0$. Thus, in order to understand $\mathscr{B}$ it suffices to study its trace $\mathscr{B} \cap \mathbb{S}^2$ with the unit sphere $\mathbb{S}^2 = \big\{(B, F, D) \in \mathbb{R}^3 : B^2 + F^2 + D^2 = 1\big\}$ because the whole bifurcation set $\mathscr{B}$ is just the positive cone with vertex at the origin over the spherical curve $\mathscr{B} \cap \mathbb{S}^2$.

The aim of the present paper is to study the bifurcations from the outer boundary of the period annulus and to prove our main result, Theorem A, it is more convenient to work in the planes $\{B = \pm 1\}$. This, on account of the aforementioned radial property, covers the whole three dimensional parameter space except for the plane $\{B = 0\}$. Theorem A is a direct consequence of Theorems 3.10 and 3.12, which are devoted to the cases $B = -1$ and $B = 1$, respectively. In order to be more precise about these results, let us follow the obvious compact notation for each case $B = \pm 1$. We define, see Figure 7, curves $\Omega_{\pm} \subset \{B = \pm 1\}$ that are proved to consist of *local bifurcation values* of the period function at the outer boundary in the sense

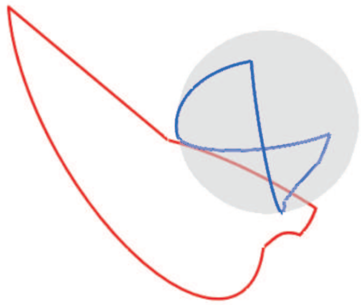

Figure 1: Local bifurcation curve Ω in blue and its stereographical projection in red.

of [18, Definition 2.4]. In addition, we introduce auxiliary curves, denoted by $\mathscr{T}_{\pm}$ and $\mathscr{R}_{\pm}$, and we show that $\{B = \pm 1\} \setminus (\Omega_{\pm} \cup \mathscr{T}_{\pm} \cup \mathscr{R}_{\pm})$ does not contain local bifurcation values at the outer boundary, i.e., it consists of *local regular values*. (Let us advance that the curves $\mathscr{T}_{\pm}$ correspond to those parameters for which the topology of the period annulus changes, see Figure 4, and $\mathscr{R}_{\pm}$ correspond to the resonant parameters in the normal forms that we use after desingularize the polycycle at its outer boundary.) We conjecture that the curves $\mathscr{T}_{\pm}$ and $\mathscr{R}_{\pm}$ contain in fact local regular values as well, i.e., that all the local bifurcation values at the outer boundary lie on $\Omega_{\pm}$.

Theorem A gathers the results in Theorems 3.10 and 3.12 explained in the previous paragraph, but for the sake of convenience and taking advantage of the radial property on the parameter space, we transfer them on the sphere $\mathbb{S}^2$. To this end, let Ω denote the closure of the union of the radial projections of the curves $\Omega_{\pm} \subset \{B = \pm 1\}$ onto the sphere $\mathbb{S}^2$. It turns out that Ω is a Jordan curve in $\mathbb{S}^2$, see Figure 1. Taking $\hat{p} := (-1, -n, -2n) \in \mathbb{R}^3$, let us denote by $\mathcal{I}_P$ (respectively, $\mathscr{D}_P$) the connected component of $\mathbb{S}^2 \setminus \Omega$ containing (respectively, not containing) the point $\hat{p}/\|\hat{p}\|$. Finally, let $\mathscr{U}$ be the union of the circle $\{B = 0\} \cap \mathbb{S}^2$ and the radial projections of the curves $\mathscr{T}_{\pm}$ and $\mathscr{R}_{\pm}$ onto the sphere $\mathbb{S}^2$. We are now in position to state the main result of this paper:

Theorem A. *Let $n \geqslant 3$ be an odd natural number and $\mu = (B, F, D)$. Let $\left\{\mathscr{L}_{n,\mu}, \mu \in \mathbb{S}^2\right\}$ be the family of vector fields in* (1) *and consider the period function of the center at the origin. Then the set $\mathbb{S}^2 \setminus (\Omega \cup \mathscr{U})$ corresponds to local regular values of the period function at the outer boundary of the period annulus. In addition,*

(a) If $\mu_0 \in \mathscr{D}_P \setminus \mathscr{U}$, then the period function of $(\mathscr{L}_{n,\mu_0})$ is monotonous decreasing near the outer boundary.

(b) If $\mu_0 \in \mathcal{I}_P \setminus \mathscr{U}$, then the period function of $(\mathscr{L}_{n,\mu_0})$ is monotonous increasing near the outer boundary.

Finally, the parameters in Ω are local bifurcation values of the period function at the outer boundary.

We expect that all the local bifurcation values at the outer boundary are inside Ω, in other words that $\mathscr{U}$ consists of local regular values. As we mentioned before, there are two types of parameters in $\mathscr{U}$, the ones where the topology of the period annulus changes, and the ones that are resonances for the normal forms of the singularities at the polycycle. We believe that the proof for the second type of parameters could be carried through with some additional analysis by applying the results obtained in [19]. The study for the first type of parameters is much more difficult and it requires the development new techniques. Of course it will still remain open the problem of how many critical periods bifurcate from the outer boundary when we perturb parameters in Ω.

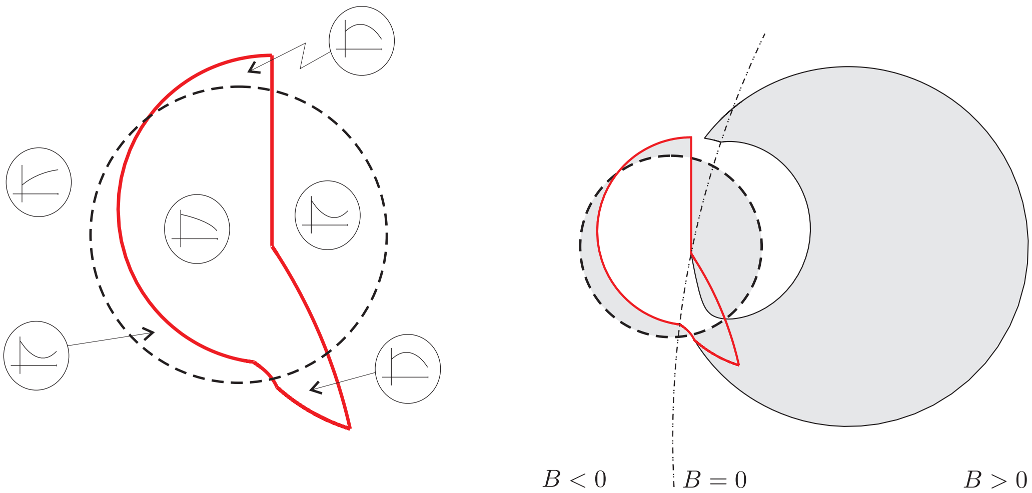

Figure 2: Conjectural bifurcation diagram of the period function, where the red curve corresponds to the bifurcation at the polycycle and the doted circle to the bifurcation at the center (left). Theorem 4.3 proves the conjecture in the complementary of the grey region (right).

The bifurcation of critical periods from the inner boundary of the period annulus, i.e., the center itself, is easier to study and it is fully understood thanks to the results in [24]. In short, [24, Theorem B] shows that this type of bifurcation values lie on the plane $\{B-F-Dn=0\}$ and that the (sharp) upper bound for the number of critical periods that bifurcate from the inner boundary is one. The proof of this result relies on the study of the ideal generated by the period constants. Obviously the most difficult part is the bifurcation from the isochrones, that for the differential system $(\mathscr{L}_{n,\mu})$ lie on the straight lines $\mu=\lambda(1,1,0)$ and $\mu=\lambda(n,n^2,1-n)$ with $\lambda\in\mathbb{R}$. Note also that the spherical trace $\mathscr{C}:=\{B-F-Dn=0\}\cap\mathbb{S}^2$ of this plane is a great circle in the sphere.

The combination of the results in [24] with the ones obtained in the present paper lead us to propose a conjecture for the global bifurcation diagram of the period function of the differential system $(\mathscr{L}_{n,\mu})$. For reader's convenience, in order to obtain a more understandable view of the conjectural bifurcation diagram we perform the stereographic projection σ of the sphere $\mathbb{S}^2$ choosing the point $\hat{p}/\|\hat{p}\|$ as "north pole". The suitability of this choice comes from the fact that, by Theorem 4.3, its corresponding differential system has a globally monotonous period function, so that the projection on the plane of all the bifurcation curves in $\mathbb{S}^2$ will be bounded. Figure 2 displays the stereographic projection of the conjectural bifurcation diagram, where the red curve is $\sigma(\Omega)$ and the doted circle is $\sigma(\mathscr{C})$.

Conjecture. *The image by the stereographic projection σ of the bifurcation diagram of the period function of $\left\{\mathscr{L}_{n,\mu},\mu\in\mathbb{S}^2\right\}$ consists in the union of the curves $\sigma(\mathscr{C})$ and $\sigma(\Omega)$, that correspond respectively to the local bifurcation values at the inner and outer boundaries. These curves split the plane in six connected components, and the period function of $(\mathscr{L}_{n,\mu})$ is either monotonous or has exactly one critical period according to Figure 2.*

We point out that this conjecture claims in particular that there are no bifurcation of critical periods from the interior of the period annulus. Besides Theorem A we have proved a number of results that give support to this conjecture. Indeed, Theorem 4.3 shows the validity of the claim concerning the monotonicity for parameters *outside* the grey region in Figure 2. Furthermore, we prove in Theorem 4.4 that there exists at least one critical period in the four components where we conjecture that there exists exactly one. Finally Proposition 4.6 shows the uniqueness of this critical period for parameters inside two segments. Let us note that the curves $\sigma(\mathscr{C})$ and $\sigma(\Omega)$ intersect in four points, see Figure 2, which correspond to the stereographic projection of the two straight lines with isochrones that we mentioned before.

The paper is organized as follows. In Section 2 we obtain the bifurcation diagram of the phase portrait of the differential system (1) for $B \neq 0$. Section 3 is devoted to the proof of Theorem A, that provides the local behaviour of the period function near the polycycle at the outer boundary of the period annulus. The most difficult cases are those in which the polycycle is unbounded and for the differential system under consideration this occurs in three different situations. The key tool is to obtain an asymptotic development of the period function near the polycycle which is uniform with respect to the parameters. To this end we rely on the results that appear in [17, 18, 20], but previously we must compactify and desingularize conveniently the polycycles. In Section 4 we study the global behaviour of the period function and we prove some partial results about monotonicity, existence and uniqueness of critical periods. Finally in the Appendix we give details concerning the radial and stereographic projections used to draw Figures 1 and 2.

The bifurcation of critical periods from the interior of the period annulus is one of the most challenging problems in the study of the period function. (Its counterpart in the context of limit cycles is the so-called *blue-sky catastrophe*.) For the quadratic Loud's centers, see [18], the existence of curves in the parameter space associated to this type of bifurcation is one of the reasons why Chicone's conjecture is still open. For the generalized Loud's centers ($\mathscr{L}_{n,\mu}$), the results in [24] seem to indicate the absence of this type of curve in case that n is odd. Our results reinforce this evidence and so the present paper can be viewed as a first step in the attempt to solve a problem that is more approachable than Chicone's conjecture. We point out that fixing the degree to be for instance $n = 3$ would allow us to go further in the proof of the conjectural bifurcation diagram, but it is not the aim of the paper to tackle a particular case. Taking n to be arbitrary complicates the proofs, while the fact that μ is three-dimensional could make the statements cumbersome. We make a special effort to avoid the latter, for instance, by performing a convenient stereographic projection of the parameter space.

2 Study of the phase portrait

In this section we study the phase portrait of the family of vector fields

$$\mathscr{L}_{n,\mu} := \big(-y + Bx^{n-1}y\big)\partial_x + \big(x + Dx^n + Fx^{n-2}y^2\big)\partial_y$$

with $\mu = (B, D, F)$ and an odd natural number $n \geqslant 3$. In what follows sometimes we avoid the subscripts in $\mathscr{L}_{n,\mu}$ for the sake of shortness.

Lemma 2.1. *The function*

$$\kappa(x) := \begin{cases} (1 - Bx^{n-1})^{-1-\frac{2F}{(n-1)B}} & \text{if } B \neq 0, \\ \exp\left(\frac{2F}{n-1}\, x^{n-1}\right) & \text{if } B = 0, \end{cases}$$

is an integrating factor for the differential system (1)*. The first integral is* $H(x,y) = A(x) + C(x)y^2$ *with*

$$A(x) := \int_0^x (s + Ds^n)\kappa(s)ds \ \text{ and } \ C(x) := \tfrac{1}{2}(1 - Bx^{n-1})\kappa(x).$$

For each value of $\mu \in \mathbb{R}^3$ and $n \in \mathbb{N}$, the vector field $\mathscr{L}_{n,\mu}$ has a center at the origin. Our goal in this section is to determine its period annulus $\mathscr{P}$ as well as its outer boundary, which is a polycycle in some compactification of $\mathbb{R}^2$. To this end we shall use the real projective plane $\mathbb{RP}^2$ covered by the charts (x, y), $(u, v) = \big(\frac{1}{x}, \frac{y}{x}\big)$ and $(\zeta, \omega) = \big(\frac{1}{y}, \frac{x}{y}\big)$. The expressions of $\mathscr{L}_{n,\mu}$ in (u, v) and (ζ, ω) coordinates are given, respectively, by

$$\mathscr{L}(u, v) = \frac{1}{u^{n-1}}\Big(-uv\big(B + u^{n-1}\big)\partial_u + \big(D + (F - B)v^2 + u^{n-1}(1 + v^2)\big)\partial_v\Big)$$

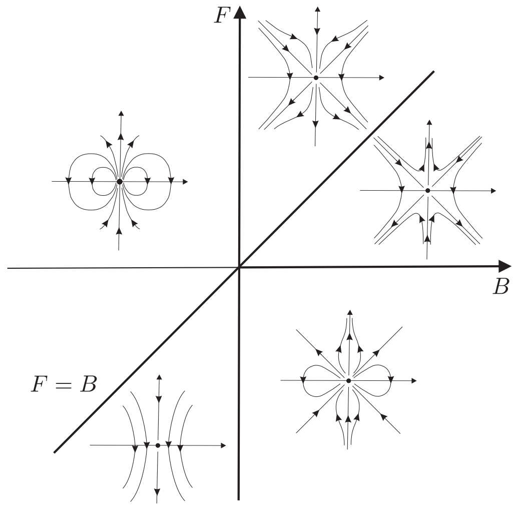

Figure 3: Phase portrait of $-\zeta^{n-1}\mathscr{L}(\zeta,\omega)$ at the origin.

and

$$\mathscr{L}(\zeta,\omega) = \frac{-1}{\zeta^{n-1}}\Big(\zeta\omega(F\omega^{n-3} + D\omega^{n-1} + \zeta^{n-1})\partial_\zeta + \big(\zeta^{n-1}(1+\omega^2) + \omega^{n-1}(F - B + D\omega^2)\big)\partial_\omega\Big).$$

Since $u^{n-1}\mathscr{L}(u,v)$ does not vanish at the origin for $D \neq 0$, to study the singular points of (1) at infinity it suffices to study $\mathscr{L}(\zeta,\omega)$. Now we proceed to study the finite and infinite singular points in some detail:

- If $B > 0$, then system (1) has two invariant straight lines at $L_\pm := \Big\{x = \pm B^{\frac{-1}{n-1}}\Big\}$. If $F(B+D) < 0$ holds additionally, then there exist two finite singular points on each invariant straight line,

  $$\Big[1, \pm\sqrt{-\tfrac{B+D}{F}}, B^{\frac{1}{n-1}}\Big] \in L_+ \text{ and } \Big[-1, \pm\sqrt{-\tfrac{B+D}{F}}, B^{\frac{1}{n-1}}\Big] \in L_-,$$

  which are nodes in case that $F > 0$ and saddles in case that $F < 0$.

- If $D < 0$, then there exist two finite singular points on $\{y = 0\}$ at $\Big[\pm 1, 0, (-D)^{\frac{1}{n-1}}\Big]$, which are centers when $B + D > 0$ and saddles when $B + D < 0$.

- There exists an infinite singular point at $[0, 1, 0]$, which is degenerate. Since one can verify that $\{\omega = 0\}$ is not a characteristic direction of $\mathscr{L}(\zeta,\omega)$ at $(0,0)$, it suffices to perform the blow up given by $\{\zeta = u\omega, \omega = \omega\}$ to desingularize it. This yields to

  $$\hat{\mathscr{L}}(u,\omega) = \frac{1}{u^{n-1}\omega}\Big(u\big(-B + u^{n-1}\big)\partial_u + \omega\big(B - F - D\omega^2 - u^{n-1}(1+\omega^2)\big)\partial_\omega\Big). \tag{2}$$

  The singular point at $(u,\omega) = (0,0)$ is a saddle if $B(B-F) > 0$ and a node if $B(B-F) < 0$. If $B > 0$, then there are two additional singular points on $\{\omega = 0\}$ at $(u,\omega) = \Big(\pm B^{\frac{1}{n-1}}, 0\Big)$, which are saddles if $F > 0$ and nodes if $F < 0$. It is easy to verify that the desingularization process yields to the bifurcation diagram displayed in Figure 3.

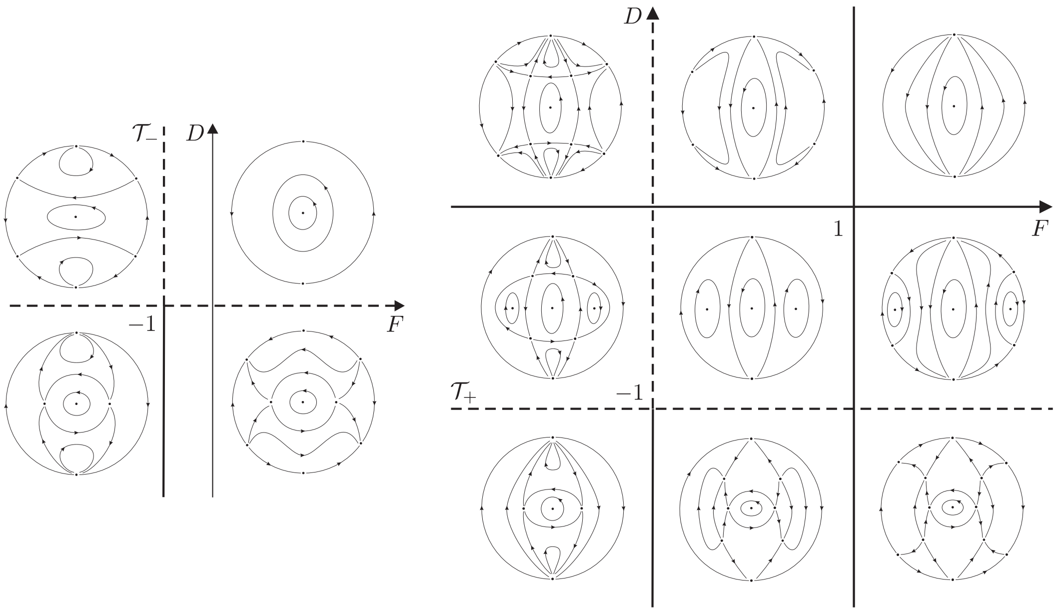

Figure 4: Bifurcation diagram of the phase portrait of the differential system (1) for $B = -1$, left, and $B = 1$, right. The bifurcations that affect the polycycle at the outer boundary of the period annulus occur only in the dotted lines $\mathcal{T}_{\pm}$.

- If $D(B-F) > 0$, then there exist four additional singular points at infinity, located at $\left[\pm 1, \sqrt{\frac{D}{B-F}}, 0\right]$, that are nodes if $BD > 0$ and saddles if $BD < 0$.

Figure 4 displays the bifurcation diagram of the phase portrait of system (1) for $B \neq 0$. One can easily obtain it by taking into account the character and distribution of the singular points as explained above, the symmetries with respect to the x and y axes, and the fact that it has a first integral which is quadratic in y. As we will see, the bifurcations that affect the polycycle at the outer boundary of $\mathscr{P}$ will play an important role in the forthcoming discussions. For $B = -1$ and $B = 1$ are given, respectively, by

$$\mathcal{T}_- := \{D = 0\} \cup \{D > 0, F = -1\} \text{ and } \mathcal{T}_+ := \{D > -1, F = 0\} \cup \{D = -1\}. \tag{3}$$

3 Bifurcation of critical periods from the polycycle

The program to carry through the study of the period function near the polycycle goes as follows. We first desingularize the critical points in the polycycle. This gives rise to a new polycycle with only hyperbolic saddles at the vertices that are orbitally linearizable. The pull-back of the vector field by the desingularization is not polynomial anymore but meromorphic. Taking advantage of the symmetries of the system under consideration it turns out that it suffices to analyse the time function associated to the passage through one of the saddles in the desingularized polycycle. At this point the key tool to prove Theorem A will be to compute the first terms in the asymptotic development of this time function. To this end we shall apply the results that appear in [17, 20]. Thus for reader's convenience we devote Subsection 3.1 to state them and introduce the related notions. Next we consider the cases $B < 0$ and $B > 0$ in Subsections 3.2 and 3.3 respectively, that can be reparameterized without loss of generality to $B = \pm 1$ on account of the radial property of $(\mathscr{L}_{n,\mu})$ explained in Section 1. We focus on those cases in which the period annulus is unbounded because this is the really complicated setting.

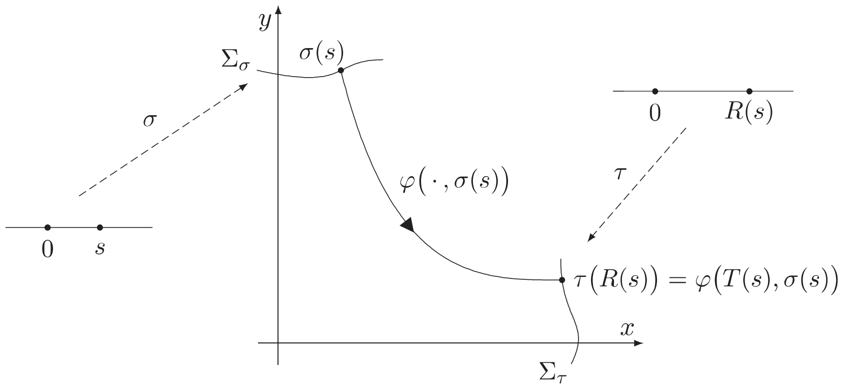

Figure 5: Definition of the time function associated to the passage through a saddle.

3.1 Technical tools

In this subsection we show how to compute the first terms in the asymptotic development of the time function associated to the passage through an orbitally linearizable saddle of a family of meromorphic vector fields.

Let W be an open set of $\mathbb{R}^k$ and consider an analytic family of meromorphic vector fields $\{X_\mu,\ \mu \in W\}$ of the form

$$X_\mu(x,y) = \frac{1}{x^p y^q}\big(xP(x,y;\mu)\partial_x + yQ(x,y;\mu)\partial_y\big), \tag{4}$$

where $p,q \in \mathbb{Z}$. We assume that P and Q are analytic functions on $V \times W$, where V is an open set of $\mathbb{R}^2$ containing the origin, and that verify $P(x,0;\mu) > 0$ and $Q(0,y;\mu) < 0$. Note then that, for each $\mu \in W$, $x^p y^q X_\mu(x,y)$ is an analytic vector field on V that has a hyperbolic saddle at the origin with *hyperbolicity ratio* given by

$$\lambda(\mu) := -\frac{Q(0,0;\mu)}{P(0,0;\mu)} > 0.$$

Let $\sigma : I \times W \longrightarrow \Sigma_\sigma$ and $\tau : I \times W \longrightarrow \Sigma_\tau$ be two analytic transverse sections to X_μ defined by

$$\sigma(s;\mu) = \big(\sigma_1(s;\mu), \sigma_2(s;\mu);\mu\big) \text{ and } \tau(s;\mu) = \big(\tau_1(s;\mu), \tau_2(s;\mu);\mu\big)$$

such that $\sigma_1(0;\mu) = 0$ and $\tau_2(0;\mu) = 0$. (Here I stands for a small interval of $\mathbb{R}$ containing 0.) We denote the Dulac and time mappings between the transverse sections Σ_σ and Σ_τ by R and T respectively. More precisely (see Figure 5), if $\varphi\big(t,(x_0,y_0);\mu\big)$ is the solution of X_μ passing through (x_0,y_0) at $t=0$, for each $s>0$ we define $R(s;\mu)$ and $T(s;\mu)$ by means of the relation

$$\varphi\big(T(s;\mu), \sigma(s);\mu\big) = \tau\big(R(s;\mu)\big). \tag{5}$$

Definition 3.1. We say that $\{X_\mu,\ \mu \in W\}$ verifies the *family linearization property* (FLP in short) if there exist an open set $U \subset \mathbb{R}^2$ containing the origin and an analytic *local* diffeomorphism $\Phi : U \times W \to V \times W$ of the form $\Phi(x,y;\mu) = \big(x + \text{h.o.t.}, y + \text{h.o.t.}, \mu\big)$ such that

$$X_\mu = \Phi_*\left(\frac{1}{f(x,y;\mu)}\big(x\partial_x - \lambda(\mu)y\partial_y\big)\right),$$

where f is an analytic function on $U \times W$. □

Remark 3.2. It is easy to show that the family of meromorphic vector fields $\{X_\mu,\, \mu \in W\}$ defined in (4) verifies FLP if it has a *generalized Darboux first integral*

$$H_\mu(x,y) = f_1(x,y;\mu)^{\beta_1(\mu)} \cdots f_k(x,y;\mu)^{\beta_k(\mu)},$$

where $f_j \in \mathcal{C}^\omega(U \times W)$ for some open set $U \subset \mathbb{R}^2$ containing the origin and $\beta_j \in \mathcal{C}^\omega(W)$. □

Definition 3.3. Let U be any open subset of $\mathbb{R}^k$. We denote by $\mathcal{I}(U)$ the set of germs of analytic functions $h(s;\mu)$ defined on $(0,\varepsilon) \times U$ for some $\varepsilon > 0$ such that

$$\lim_{s\to 0} h(s;\mu) = 0 \text{ and } \lim_{s\to 0} s\,\frac{\partial h(s;\mu)}{\partial s} = 0$$

uniformly (on μ) on every compact subset of U. □

Definition 3.4. The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{\alpha-1}-1}{\alpha-1} & \text{if } \alpha \neq 1, \\ \log s & \text{if } \alpha = 1, \end{cases}$$

is called the *Écalle-Roussarie compensator.* □

In order to simplify the expressions that appear henceforth we introduce the functions

$$L(u;\mu) := \exp\left(\int_{\sigma_2(0)}^{u} \left(\frac{P(0,y)}{Q(0,y)} + \frac{1}{\lambda}\right)\frac{dy}{y}\right),$$
$$M(u;\mu) := \exp\left(\int_{0}^{u} \left(\frac{Q(x,0)}{P(x,0)} + \lambda\right)\frac{dx}{x}\right),$$

and the cover of the parameter space W given by

$$W_1 := \left\{\mu \in W : \lambda(\mu) < \tfrac{p}{q}\right\},$$
$$W_2 := \left\{\mu \in W : \lambda(\mu) > \tfrac{p}{q}\right\},$$
$$W_3 := \left\{\mu \in W : \lambda(\mu) \in \left(\tfrac{p}{q+1}, \tfrac{p+1}{q}\right)\right\}.$$

The next result is proved in [20, Theorem A].

Theorem 3.5 (First order development)**.** *Let $\{X_\mu,\, \mu \in W\}$ be the family of vector fields defined in* (4) *and assume that it verifies the FLP. Let T be the time function associated to the transverse sections Σ_σ and Σ_τ as introduced in* (5)*. Then the following holds:*

(a) If $\mu \in W_1$ then $T(s;\mu) = s^p\big(\Delta_1(\mu) + \mathcal{I}(W_1)\big)$, where

$$\Delta_1(\mu) = \sigma_1'(0)^p \sigma_2(0)^{\frac{p}{\lambda}} \int_{\sigma_2(0)}^{0} \frac{L(y)^p y^{q-\frac{p}{\lambda}}}{Q(0,y)}\frac{dy}{y}.$$

(b) If $\mu \in W_2$ then $T(s;\mu) = s^{\lambda q}\big(\Delta_2(\mu) + \mathcal{I}(W_2)\big)$, where

$$\Delta_2(\mu) = \left(\sigma_1'(0)^\lambda \sigma_2(0) L(0)^\lambda\right)^q \int_{0}^{\tau_1(0)} \frac{M(x)^q x^{p-\lambda q}}{P(x,0)}\frac{dx}{x}.$$

(c) *If $\mu \in W_3$ then $T(s;\mu) = s^{\lambda q}\big(\Delta_3(\mu)\omega(s;p+1-\lambda q) + \Delta_4(\mu) + \mathcal{I}(W_3)\big)$, where Δ_3 and Δ_4 are analytic on W_3. Furthermore, if $\lambda(\mu_0) = p/q$ then*

$$\Delta_3(\mu_0) = -\frac{\big(\sigma_1'(0)^\lambda L(0)^\lambda \sigma_2(0)\big)^q}{P(0,0)}.$$

Let us clarify that, for the sake of simplicity in the exposition, in the statement we write $\Delta(\mu) + \mathcal{I}(W)$ meaning $\Delta(\mu) + h(s;\mu)$ with $h \in \mathcal{I}(W)$. In what follows we shall make this abuse of notation when there is no danger of confusion. For the case $p = 0$ we shall need an additional term in the asymptotic development. In order to treat this case we introduce the cover of the parameter space given by

$$\begin{aligned} U_1 &:= \left\{\mu \in W : \lambda(\mu) > \tfrac{1}{q}\right\}, \\ U_2 &:= \left\{\mu \in W : \lambda(\mu) < \tfrac{1}{q}\right\}, \\ U_3 &:= \left\{\mu \in W : \lambda(\mu) \in \left(\tfrac{1}{q+1}, \tfrac{2}{q}\right)\right\}. \end{aligned}$$

The following is the main result in [17].

Theorem 3.6 (Second order development)**.** *Let $\{X_\mu,\ \mu \in W\}$ be the family of vector fields defined in (4) taking $p = 0$ and assume that it verifies the FLP. Let T be the time function associated to the transverse sections Σ_σ and Σ_τ as introduced in (5). Denoting*

$$\Delta_1(\mu) = \int_{\sigma_2(0)}^0 \frac{y^{q-1}}{Q(0,y)}\,dy,$$

the time function $T(s;\mu)$ verifies the following:

(a) *If $\mu \in U_1$ then $T(s;\mu) = \Delta_1(\mu) + s\big(\Pi_1(\mu) + \mathcal{I}(U_1)\big)$, where*

$$\Pi_1(\mu) = -\frac{\sigma_2'(0)\,\sigma_2(0)^{q-1}}{Q(0,\sigma_2(0))} + \sigma_1'(0)\sigma_2(0)^{1/\lambda}\int_0^{\sigma_2(0)} \frac{Q_x(0,y)L(y)y^{q-1/\lambda}}{Q(0,y)^2}\frac{dy}{y}.$$

(b) *If $\mu \in U_2$ then $T(s;\mu) = \Delta_1(\mu) + s^{\lambda q}\big(\Pi_2(\mu) + \mathcal{I}(U_2)\big)$, where*

$$\Pi_2(\mu) = \sigma_1'(0)^{\lambda q}\sigma_2(0)^q L(0)^{\lambda q}\left\{\frac{\tau_1(0)^{-\lambda q}}{qQ(0,0)} + \int_0^{\tau_1(0)}\left(\frac{M(x)^q}{P(x,0)} - \frac{M(0)^q}{P(0,0)}\right)\frac{dx}{x^{\lambda q+1}}\right\}.$$

(c) *If $\mu \in U_3$ then $T(s;\mu) = \Delta_1(\mu) + s\big(\Pi_3(\mu)\omega(s;\lambda q) + \Pi_4(\mu) + \mathcal{I}(U_3)\big)$, where Π_3 and Π_4 are analytic on U_3. Furthermore, if $\lambda(\mu_0) = 1/q$ then*

$$\Pi_3(\mu_0) = -q\sigma_1'(0)\sigma_2(0)^q L(0)\frac{Q_u(0,0)}{P(0,0)^2}.$$

We conclude this subsection with a technical result that we shall use henceforth in some computations and that refers to the Gamma function, which is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt \text{ for any } z \in \mathbb{C} \text{ with } \mathrm{Re}(z) > 0.$$

Lemma 3.7. *Let w and z be any complex numbers with strictly positive real part. Then*

$(a)\ \displaystyle\int_0^1 u^{z-1}(1-u)^{w-1}du=\int_0^\infty \frac{u^{z-1}du}{(1+u)^{z+w}}=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$,

$(b)\ \displaystyle\int_0^1 u^{z-1}\big((1-u)^{w-1}-1\big)\,du=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}-\frac{1}{z}$, *provided* $\mathrm{Re}(z)>-1$,

$(c)\ \displaystyle\int_0^1 \left(\left(\frac{1-(1-u)^{n-1}}{n-1}\right)^{z-1}-u^{z-1}\right)du=\frac{1}{(n-1)^z}\frac{\Gamma(z)\Gamma\big(\frac{1}{n-1}\big)}{\Gamma\big(z+\frac{1}{n-1}\big)}-\frac{1}{z}$, *provided* $\mathrm{Re}(z)>-1$ *and* $n>1$.

Proof. The equalities in (a) are well known, see for instance (6.2.1) and (6.2.2) in [1]. The equality in (b) follows directly from (a) in case that $\mathrm{Re}(z)>0$. On account of this, the equality on $\mathrm{Re}(z)>-1$ will follow by analytic continuation. Indeed, if $\mathrm{Re}(z)>-1$ then the integral on the left hand side is convergent because $u\longmapsto\frac{(1-u)^{w-1}-1}{u}$ is analytic at $u=0$. In addition, for each fixed w, the function on the right hand side is a (univalued) holomorphic function on $\mathrm{Re}(z)>-1$ because, see (6.1.34) in [1], $z\Gamma(z)\longrightarrow 1$ as z tends to 0. This proves (b). Let us turn finally to the assertion in (c). For convenience we will first prove it on $\mathrm{Re}(z)>0$. In this case,

$$\begin{aligned}\int_0^1 \left(\left(\frac{1-(1-u)^{n-1}}{n-1}\right)^{z-1}-u^{z-1}\right)du&=\frac{1}{(n-1)^z}\int_0^1(1-v)^{z-1}v^{\frac{1}{n-1}-1}dv-\left.\frac{u^z}{z}\right|_0^1\\&=\frac{1}{(n-1)^z}\frac{\Gamma(z)\Gamma\big(\frac{1}{n-1}\big)}{\Gamma\big(z+\frac{1}{n-1}\big)}-\frac{1}{z},\end{aligned}$$

where in the first equality we perform the change of variables $v=(1-u)^{n-1}$ and in the second one we use (a). As before the equality on $\mathrm{Re}(z)>-1$ follows by analytic continuation. To show this we rewrite the integrand as

$$\left(\frac{1-(1-u)^{n-1}}{n-1}\right)^{z-1}-u^{z-1}=u^{z-1}\left(\left(\frac{1-(1-u)^{n-1}}{(n-1)u}\right)^{z-1}-1\right)$$

and we observe that the function $u\longmapsto\left(\frac{1-(1-u)^{n-1}}{(n-1)u}\right)^{z-1}-1$ is analytic and vanishes at $u=0$. ■

3.2 The case B < 0

We treat first those parameters for which the center is global, i.e., setting $B=-1$, we consider $\mu:=(D,F)$ inside $\{D>0,F>-1\}$, see Figure 4. In the next result $T(s;\mu)$ is the period of the periodic orbit of system (1) passing through the point $(1/s,0)\in\mathbb{R}^2$ and we provide its asymptotic development for $s\approx 0$. In its statement we denote

$$\mathscr{R}_-:=\bigcup_{i\geqslant 0}\left\{F=-1+i\frac{n-1}{2}\right\},\tag{6}$$

which corresponds to the resonant parameters in the normal form of the saddle at infinity that we obtain after desingularize the polycycle. Let us advance that the result follows by applying Theorem 3.5 and point out that the case involving the Écalle-Roussarie compensator is missing because it would cover $F=n-2$, which is a resonant parameter

Lemma 3.8. *Let us define* $W_1=\{D>0,F>n-2\}\setminus\mathscr{R}_-$ *and* $W_2=\big\{D>0,F\in(-1,n-2)\big\}\setminus\mathscr{R}_-$.

(a) *If* $\mu\in W_1$ *then* $T(s;\mu)=s^{n-1}\big(\Delta_1(\mu)+\mathcal{I}(W_1)\big)$, *where* Δ_1 *is analytic and strictly positive on* W_1.

(b) *If* $\mu\in W_2$ *then* $T(s;\mu)=s^{F+1}\big(\Delta_2(\mu)+\mathcal{I}(W_2)\big)$, *where* Δ_2 *is analytic and strictly positive on* W_2.

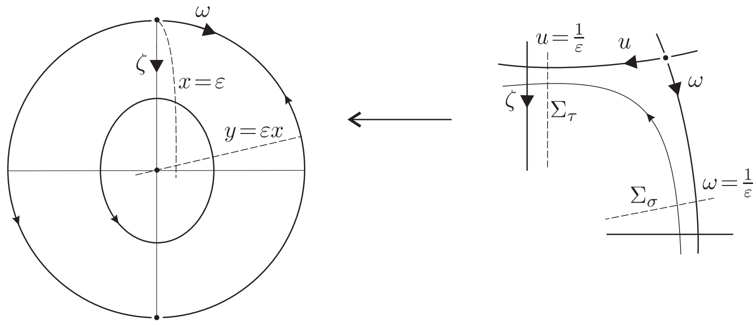

Figure 6: Desingularization and transverse sections in Lemma 3.8.

Proof. The polycycle at the outer boundary of the period annulus is the line at infinity and in order to study it we use the chart $(\zeta,\omega)=\big(\frac{1}{y},\frac{x}{y}\big)$, see Section 2. It has only a singular point at $(\zeta,\omega)=(0,0)$, which is degenerated. To desingularize it we perform the blow up $\{\zeta=u\omega,\omega=\omega\}$ and we obtain (2). This is a family of meromorphic vector fields of the form (4) with $\{p=n-1,q=1\}$ and

$$P(u,\omega)=1+u^{n-1}\ \text{ and } Q(u,\omega)=-1-F-D\omega^2-u^{n-1}(1+\omega^2),$$

so that $\lambda=-\frac{Q(0,0)}{P(0,0)}=F+1$.

In order to apply Theorem 3.5 it is first necessary to check that $X_\mu(u,\omega)$ with $\mu\in\{D>0,F>-1\}\setminus\mathscr{R}_-$ verifies the FLP, see Definition 3.1. Taking $\{x=\frac{1}{u},y=\frac{1}{u\omega}\}$ into account, by applying Lemma 2.1 we know that $\widehat{H}(u,\omega):=A(\frac{1}{u})+C(\frac{1}{u})\frac{1}{(u\omega)^2}$ is a first integral of $X_\mu(u,\omega)$. Some easy computations show that

$$\frac{d}{du}A(1/u)=u^{-2F-3}f_0(u^{n-1})\ \text{ with } f_0(u):=-(u+D)(u+1)^{-1+\frac{2F}{n-1}}.$$

Clearly f_0 is an analytic function at $u=0$. If $f_0(u)=\sum_{i\geqslant 0}a_iu^i$ is its Taylor's development at $u=0$ and we set $\alpha=2F+3$ for shortness, then $\frac{d}{du}A(\frac{1}{u})=\sum_{i\geqslant 0}a_iu^{(n-1)i-\alpha}$. Accordingly

$$A(1/u)=\sum_{i\geqslant 0}\frac{a_i}{(n-1)i-\alpha+1}u^{(n-1)i-\alpha+1},\ \text{ provided } \tfrac{\alpha-1}{n-1}+1\notin\mathbb{N}.$$

Since this condition writes as $F\notin\mathscr{R}_-$, we can assert that $A(\frac{1}{u})=u^{1-\alpha}f_1(u^{n-1})$ with f_1 being an analytic function at $u=0$. On the other hand, $C(1/u)=u^{-2F}f_2(u^{n-1})$ with $f_2(u)=(u+1)^{\frac{2F}{n-1}}$, which is analytic at $u=0$ as well. Therefore

$$\widehat{H}(u,\omega)=\frac{\omega^2f_1(u^{n-1})+f_2(u^{n-1})}{u^{2(F+1)}\omega^2}$$

is a generalized Darboux first integral for $X_\mu(u,\omega)$ with $\mu\in\{D>0,F>-1\}\setminus\mathscr{R}_-$ and this, on account of Remark 3.2, shows that the family of meromorphic vector fields under consideration verifies the FLP.

For convenience we study, see Figure 6, the time function associated to the passage from the transverse sections $\Sigma_\sigma=\{y=\varepsilon x\}$ to $\Sigma_\tau=\{x=\varepsilon\}$, where $\varepsilon>0$. More precisely, taking (u,ω)-coordinates, we parametrize Σ_σ and Σ_τ with $\sigma(s)=(s,1/\varepsilon)$ and $\tau(s)=(1/\varepsilon,\varepsilon s)$, respectively. Define $\hat{\mu}=(D,F,\varepsilon)$ and let $\hat{T}(s;\hat{\mu})$ be the time that spends the solution of (2) passing through the point $\sigma(s)\in\Sigma_\sigma$ to reach Σ_τ. Some computations show that

$$L(u)=\left(\frac{F+1+Du^2}{F+1+D\varepsilon^{-2}}\right)^{\frac{1}{2(F+1)}}\ \text{ and } M(u)=\big(1+u^{n-1}\big)^{\frac{F}{n-1}}.$$

By applying Theorem 3.5 we can assert that if $\mu \in W_1$ then $\hat{T}(s;\hat{\mu}) = s^{n-1}\big(\hat{\Delta}_1(\hat{\mu}) + \mathcal{I}(W_1 \times V)\big)$, where V is a neighbourhood of $\varepsilon = 0$ and

$$\hat{\Delta}_1(\hat{\mu}) = \int_0^{1/\varepsilon} \frac{u^{-\frac{n-1}{\lambda}}}{F+1+Du^2}\left(\frac{F+1+Du^2}{D+\varepsilon(F+1)}\right)^{\frac{n-1}{2(F+1)}} du.$$

Since the differential system (1) is symmetric with respect to both coordinate axes, it turns out that $T(s;\mu) = 4\lim_{\varepsilon\to 0}\hat{T}(s;\hat{\mu})$. In fact, following exactly the same approach as in the proof of [18, Theorem 3.6], one can show that $T(s;\mu) = s^{n-1}\big(\Delta_1(\mu) + \mathcal{I}(W_1)\big)$ with

$$\Delta_1(\mu) = 4\lim_{\varepsilon\to 0}\hat{\Delta}(\hat{\mu}) = \frac{4}{D}\int_0^{\infty}\left(\frac{F+1}{D} + u^2\right)^{\frac{n-1}{2(F+1)}-1} u^{-\frac{n-1}{\lambda}}\, du.$$

This proves (a) because it is clear that Δ_1 is strictly positive on W_1.

On the other hand, by applying Theorem 3.5 again, if $\mu \in W_2$ then $\hat{T}(s;\hat{\mu}) = s^{\lambda}\big(\hat{\Delta}_2(\hat{\mu}) + \mathcal{I}(W_2 \times V)\big)$, where V is a neighbourhood of $\varepsilon = 0$ and

$$\hat{\Delta}_2(\hat{\mu}) = \sqrt{\frac{F+1}{D+\varepsilon(F+1)}}\int_0^{1/\varepsilon}\frac{(1+u^{n-1})^{\frac{F}{n-1}-1}}{u^{2-n+\lambda}}\, du.$$

As before, $T(s;\mu) = s^{\lambda}\big(\Delta_2(\mu) + \mathcal{I}(W_2)\big)$ with

$$\Delta_2(\mu) = 4\lim_{\varepsilon\to 0}\hat{\Delta}_2(\hat{\mu}) = 4\sqrt{\frac{F+1}{D}}\int_0^{\infty}\frac{(1+u^{n-1})^{\frac{F}{n-1}-1}}{u^{2-n+\lambda}}\, du,$$

and this completes the proof of the result because Δ_2 is strictly positive on W_2. ■

Let us consider now the case $\{D > 0, F < -1\}$ and recall that we set $B = -1$ without lost of generality. For these parameter values, see Figure 4, the outer boundary of the period annulus is a polycycle with four singularities, all of them being hyperbolic saddles at infinity. In the next result $T(s;\mu)$ denotes the period of the periodic orbit of system (1) passing through the point $(1/\sqrt{s}, 0) \in \mathbb{R}^2$.

Lemma 3.9. *Let us define the subsets* $U_1 = \big\{D > 0, F \in \big(-\frac{n+1}{2}, -1\big)\big\}$, $U_2 = \big\{D > 0, F < -\frac{n+1}{2}\big\}$ *and* $U_3 = \big\{D > 0, F \in \big(-\frac{n+3}{2}, -\frac{n+3}{4}\big)\big\}$.

- (a) *If* $\mu \in U_1$ *then* $T(s;\mu) = \Delta_1(\mu) + s\big(\Pi_1(\mu) + \mathcal{I}(U_1)\big)$, *where* Δ_1 *and* Π_1 *are analytic on* U_1. *In addition* Π_1 *is strictly positive.*
- (b) *If* $\mu \in U_2$ *then* $T(s;\mu) = \Delta_1(\mu) + s^{-\frac{n-1}{2(F+1)}}\big(\Pi_2(\mu) + \mathcal{I}(U_2)\big)$, *where* Δ_1 *and* Π_2 *are analytic on* U_2. *Moreover,* $\Pi_2(\mu) = 0$ *with* $\mu \in U_2$ *if, and only, if* $F = -n$ *and* Π_2 *is strictly positive (respectively, negative) for* $F \in \big(-n, -\frac{n+1}{2}\big)$ *(respectively,* $F < -n$*).*
- (c) *If* $\mu \in U_3$ *then* $T(s;\mu) = \Delta_1(\mu) + s\big(\Pi_3(\mu)\omega(s; -\frac{n-1}{2(F+1)}) + \Pi_4(\mu) + \mathcal{I}(U_3)\big)$, *where* Δ_1, Π_3 *and* Π_4 *are analytic on* U_3. *Moreover, if* $\mu_0 = (D_0, F_0)$ *with* $F_0 = -\frac{n+1}{2}$, *then* $\Pi_3(\mu_0) > 0$.

Proof. On account of Lemma 2.1 system (1) has a first integral of the form $H(x,y) = A(x) + C(x)y^2$. In addition, see Figure 4, the outer boundary of the period annulus is inside the level curve $\{H = r\}$ where

$$r := \lim_{x\longrightarrow\infty} A(x) = \int_0^{\infty}\frac{s + Ds^n}{(1+s^{n-1})^{1-\frac{2F}{n-1}}} ds.$$

The polycycle is composed by the two segments of the line at infinity joining the hyperbolic saddles, together with the two connected components of the curve given by the affine equation $y^2 + \frac{A(x)-r}{C(x)} = 0$. Since n is odd it follows that $x \longmapsto \frac{A(x)-r}{C(x)}$ is an even function, and consequently we can write $\frac{A(x)-r}{C(x)} = \psi(x^2)$, where ψ is an analytic function. Moreover, taking $\lim_{x\to\infty} \frac{A'(x)}{x^{2F+1}} = D$ into account, one can easily show by applying L'Hôpital's rule that $\lim_{x\longrightarrow\infty} \frac{A(x)-r}{x^{2(F+1)}} = \frac{D}{2(F+1)}$. Thus, since $\lim_{x\to\infty} \frac{C(x)}{x^{2F}} = \frac{1}{2}$, we conclude that

$$\lim_{x\longrightarrow\infty} \frac{\psi(x^2)}{x^2} = \frac{D}{F+1}. \tag{7}$$

On the other hand, the implicit differentiation in the equality that defines ψ yields

$$\psi'(x^2) = \frac{1 + Dx^{n-1} - Fx^{n-3}\psi(x^2)}{1 + x^{n-1}}.$$

Taking this equality into account, some long but straightforward computations show that the local coordinate transformation given by $\left\{z = -\frac{y^2+\psi(x^2)}{x^2}, w = \frac{1}{x^2}\right\}$ and the reversion of time brings system (1) to a family of vector fields of the form (4) with $\{p = 0, q = \frac{n-1}{2}\}$ and

$$P(z, w) = -2(F + 1 + w^{\frac{n-1}{2}})\sqrt{-z - w\psi(1/w)} \text{ and } Q(z, w) = -2(1 + w^{\frac{n-1}{2}})\sqrt{-z - w\psi(1/w)}.$$

These are well defined functions on $w = 0$ because, on account of (7),

$$P(z, 0) = -2(F + 1)\sqrt{\frac{-D}{F+1} - z} \text{ and } Q(z, 0) = -2\sqrt{\frac{-D}{F+1} - z}, \tag{8}$$

so that $\lambda = -\frac{Q(0,0)}{P(0,0)} = \frac{-1}{F+1}$. Taking a small parameter $\varepsilon > 0$, we consider the transverse sections to the polycycle given by $\Sigma_\sigma = \{x = \sqrt{\varepsilon}\}$ and $\Sigma_\tau = \{y = 0\}$, parameterized respectively with

$$s \longmapsto \left(\sqrt{\varepsilon}, \sqrt{-\psi(\varepsilon) - s}\right) \text{ and } s \longmapsto \left(1/\sqrt{s}, 0\right).$$

Using (z, w)-coordinates they write as $\sigma(s) = (s/\varepsilon, 1/\varepsilon)$ and $\tau(s) = \left(-s\psi(1/s), s\right)$, respectively. Define $\hat{\mu} = (D, F, \varepsilon)$ and let $\hat{T}(s; \hat{\mu})$ be the time that spends the solution passing through $\sigma(s) \in \Sigma_\sigma$ to reach Σ_τ.

Since one can verify that $\widehat{H}(z, w) = \frac{w^{F+1}(1+\omega^{\frac{n-1}{2}})^{\frac{-2F}{n-1}}}{z}$ is a first integral for $X_\mu(z, \omega)$, from Remark 3.2 it follows that the family of meromorphic vector fields under consideration satisfies the FLP. We can thus apply Theorem 3.6. To this end let us first note that

$$L(u) = \left(\frac{u^{\frac{n-1}{2}} + 1}{\varepsilon^{-\frac{n-1}{2}} + 1}\right)^{\frac{2F}{1-n}} \text{ and } M(u) = 1.$$

If $\mu \in U_1$ then $\hat{T}(s; \hat{\mu}) = \hat{\Delta}_1(\hat{\mu}) + s\hat{\Pi}_1(\hat{\mu}) + s\mathcal{I}(U_1 \times V)$, where V is a neighbourhood of $\varepsilon = 0$. In addition, setting $q = \frac{n-1}{2}$ for the sake of shortness, some computations show that

$$\hat{\Pi}_1(\hat{\mu}) = -\frac{1}{4}\int_0^{1/\varepsilon} \frac{(1 + w^q)^{-\frac{F}{q}-1} w^{q-\frac{1}{\lambda}}}{\left(-w\psi\left(\frac{1}{w}\right)\right)^{\frac{3}{2}} (1 + \varepsilon^q)^{-\frac{F}{q}}} \frac{dw}{w}.$$

Due to the symmetries of system (1) and the fact that we reversed the time, by construction we have that $T(s; \mu) = -4 \lim_{\varepsilon\to 0} \hat{T}(s; \hat{\mu})$. In fact one can show that $T(s; \mu) = \Delta_1(\mu) + s\Pi_1(\mu) + s\mathcal{I}(U_1)$ with

$$\Delta_1(\mu) = -4 \lim_{\varepsilon\to 0} \hat{\Delta}_1(\hat{\mu}) \text{ and } \Pi_1(\mu) = -4 \lim_{\varepsilon\to 0} \hat{\Pi}_1(\hat{\mu}) = \int_0^{+\infty} \frac{(1 + w^q)^{-\frac{F}{q}-1} w^{q-\frac{1}{\lambda}}}{\left(-w\psi\left(\frac{1}{w}\right)\right)^{\frac{3}{2}}} \frac{dw}{w},$$

which is clearly positive. This proves the statement in (a).

On the other hand if $\mu \in U_2$ then $\hat{T}(s;\hat{\mu}) = \hat{\Delta}_1(\hat{\mu}) + s^{\lambda q}\hat{\Pi}_2(\hat{\mu}) + s^{\lambda q}\mathcal{I}(U_2 \times V)$, where V is again a neighbourhood of $\varepsilon = 0$. In order to compute $\hat{\Pi}_2$ let us first note that, from (7), $\tau(0) = \big(\frac{-D}{F+1}, 0\big)$. Therefore, taking (8) into account as well, we obtain

$$\begin{aligned}\hat{\Pi}_2(\hat{\mu}) &= \frac{-1}{2(1+\varepsilon^q)^{\lambda F}}\left\{\frac{1}{q}\left(\frac{-D}{F+1}\right)^{-(\lambda q+\frac{1}{2})} + \frac{1}{\sqrt{-D(F+1)}}\int_0^{\frac{-D}{F+1}}\left(\frac{1}{\sqrt{1+\frac{F+1}{D}z}}-1\right)\frac{dz}{z^{\lambda q+1}}\right\}\\ &= \frac{-\sqrt{\pi}}{2(1+\varepsilon^q)^{\lambda F}}\left(\frac{-D}{1+F}\right)^{-\lambda q+\frac{1}{2}}\frac{\Gamma(-\lambda q)}{\Gamma\left(\frac{1}{2}-\lambda q\right)},\end{aligned}$$

where in the last equality we use relation (b) in Lemma 3.7 with an appropriate rescaling in the variable z. Accordingly, exactly as in the previous case, it follows that $T(s;\mu) = \Delta_1(\mu) + s^{\lambda q}\Pi_2(\mu) + s^{\lambda q}\mathcal{I}(U_2)$ with

$$\Pi_2(\mu) = -4\lim_{\varepsilon\to 0}\hat{\Pi}_2(\hat{\mu}) = 2\sqrt{\pi}\left(\frac{-D}{1+F}\right)^{-\lambda q+\frac{1}{2}}\frac{\Gamma(-\lambda q)}{\Gamma\left(\frac{1}{2}-\lambda q\right)}.$$

Note that the Gamma function is a non-vanishing meromorphic function with poles at the non-positive integers. Consequently $\Pi_2(\mu) = 0$ with $\mu \in U_2$ if, and only, if $\lambda q = \frac{1}{2}$, i.e., $F = -n$. One can also check that $\Pi_2(\mu) < 0$ for $F < -n$ and $\Pi_2(\mu) > 0$ for $-n < F < -\frac{n+1}{2}$. This proves (b).

Finally, if $\mu \in U_3$ then we have that $\hat{T}(s;\hat{\mu}) = \hat{\Delta}_1(\hat{\mu}) + s\big(\hat{\Pi}_3(\hat{\mu})\omega(s;\lambda q) + \hat{\Pi}_4(\hat{\mu}) + \mathcal{I}(U_3\times V)\big)$, where V is again a neighbourhood of $\varepsilon = 0$. Moreover, if $\hat{\mu}_0 = \big(D_0, -\frac{n+1}{2}, \varepsilon\big)$ then some computations show that

$$\hat{\Pi}_3(\hat{\mu}_0) = \frac{-1}{4D_0}\sqrt{\frac{n-1}{2D_0}}\big(1+\varepsilon^{\frac{n-1}{2}}\big)^{-\frac{n+1}{n-1}}.$$

Thus, exactly as before, $T(s;\mu) = \Delta_1(\mu) + s\big(\Pi_3(\mu)\omega(s;\lambda q) + \Pi_4(\mu) + \mathcal{I}(U_3)\big)$, where $\Pi_i(\mu) = -4\lim_{\varepsilon\to 0}\hat{\Pi}_i(\hat{\mu})$. Consequently $\Pi_3(\mu_0) = \frac{1}{D_0}\sqrt{\frac{n-1}{2D_0}}$, which is positive because $D_0 > 0$. This shows (c) and completes the proof of the result. ■

The following result summarizes the results about the behaviour of the period function near the polycycle that we have proved so far. Recall that we consider the differential system (1) with $B = -1$, i.e.,

$$\begin{cases} \dot{x} = -y - x^{n-1}y, \\ \dot{y} = x + Dx^n + Fx^{n-2}y^2, \end{cases} \tag{9}$$

Define $\Omega_- = \big\{D = 0, F > -n\big\} \cup \big\{D > 0, F = -n\big\}$. Recall as well that $\mathscr{T}_-$ and $\mathscr{R}_-$ are given in (3) and (6), respectively. The reader is referred to [17] for the definition of *local regular value* of the period function at the outer boundary. Roughly speaking these are the parameters for which no critical periods bifurcate from the polycycle. A parameter which is not a local regular value at the outer boundary is called a *local bifurcation value* at the outer boundary.

Theorem 3.10. *Consider the period function of the center at the origin of the differential system* (9)*. Then the set* $\mathbb{R}^2 \setminus (\Omega_- \cup \mathscr{R}_- \cup \mathscr{T}_-)$ *corresponds to local regular values at the outer boundary of the period annulus. For these parameters the period function is monotonous near the outer boundary and the corresponding character is shown in Figure 7. Finally* Ω_- *are local bifurcation values at the outer boundary.*

Proof. Note first of all that if $\mu^\star = (D^\star, F^\star) \notin \Omega_- \cup \mathscr{R}_- \cup \mathscr{T}_-$ then either $D^\star < 0$ or it belongs to some of the sets W_i (respectively, U_i) in the statement of Lemma 3.8 (respectively, Lemma 3.9). Let us postpone the case $D^\star < 0$ and suppose for instance that $\mu^\star \in W_1$ (the other cases follow exactly the same way).

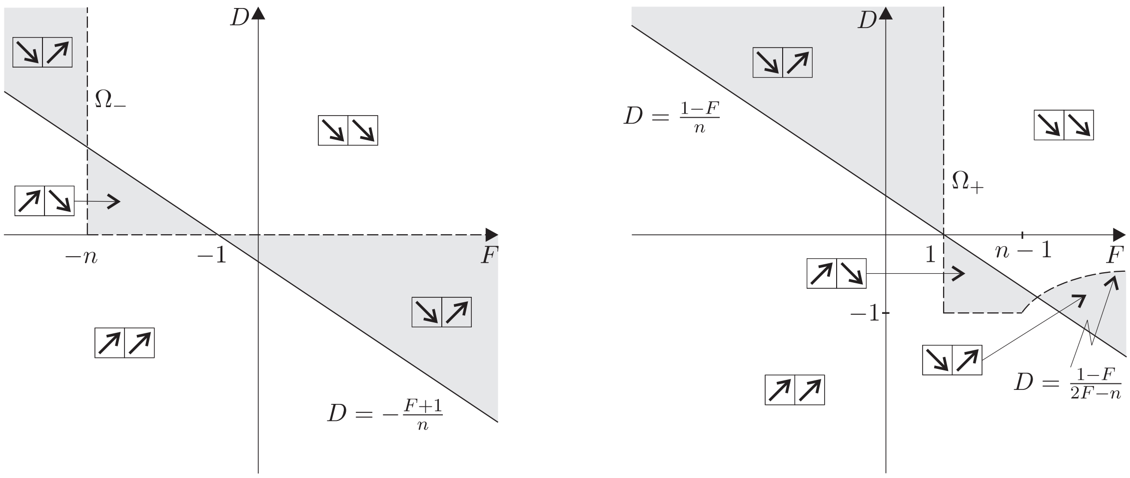

Figure 7: On the left (respectively, right), local behaviour of the period function at the boundaries of the period annulus for system (9) (respectively, (11)) and parameters (F, D) not in $\Omega_- \cup \mathscr{R}_- \cup \mathcal{T}_-$ (respectively, $\Omega_+ \cup \mathscr{R}_+ \cup \mathcal{T}_+$). The left and right arrows in each rectangle stand, respectively, for the monotonicity near the center and the polycycle. The grey regions consists of parameters with at least one critical period.

In this case by (a) in Lemma 3.8 we can assert that if $\mu \approx \mu^\star$ then $T(s;\mu) = s^{n-1}\big(\Delta_1(\mu) + f(s;\mu)\big)$ with $f \in \mathcal{I}(W_1)$ and $\Delta_1(\mu^\star) > 0$. Taking Definition 3.3 into account we thus obtain

$$\frac{1}{s^{n-2}}\frac{\partial T(s;\mu)}{\partial s} = \Delta_1(\mu) + s\frac{\partial f(s;\mu)}{\partial s} \longrightarrow \Delta_1(\mu^\star) \text{ as } (s,\mu) \longrightarrow (0,\mu^\star).$$

Hence there exist a neighbourhood U of $\mu^\star$ and $\varepsilon > 0$ such that $T_s(s;\mu) > 0$ for all $s \in (0,\varepsilon)$ and $\mu \in U$. This shows that no critical period bifurcates from the outer boundary for $\mu \approx \mu^\star$. In addition, since $s \searrow 0$ corresponds to periodic orbits approaching to the polycycle, the period function of system (9) with $\mu = (D^\star, F^\star)$ is monotonous decreasing near the outer boundary.

Let us suppose finally that $D^\star < 0$. In this case, see Figure 4, the period annulus $\mathscr{P}$ is bounded and the singularities at its outer boundary are two (finite) hyperbolic saddles. We shall take advantage of the symmetry of the system with respect to both coordinate axes. Thus, let $(-\hat{y}, \hat{y})$ the the projection of $\mathscr{P}$ on the y-axis and, for $s > 0$, let $\hat{T}(s;\mu)$ be the time that spends the periodic orbit of (9) for going from the point $(0, -\hat{y}+s)$ to $(0, \hat{y}-s)$. Note that the period of the periodic orbit is precisely $2\hat{T}(s;\mu)$. By applying [20, Theorem A] it follows that if $\lambda_1(\mu)$ is the positive eigenvalue of the saddle, then

$$\hat{T}(s;\mu) = -\frac{1}{\lambda_1(\mu)}\log s + \Delta_4(\mu) + f(s;\mu),$$

where Δ_4 is analytic and $f \in \mathcal{I}(U)$ for some neighbourhood U of $\mu^\star$. Hence $s\hat{T}_s(s;\mu)$ tends to $-1/\lambda(\mu^\star)$ as $(s,\mu) \longrightarrow (0,\mu^\star)$. This proves that $\mu^\star$ is a local regular value and that the period function is monotonous increasing near the polycycle. Finally the last assertion follows from the fact that in any neighbourhood of a parameter $\mu \in \Omega_-$ there are two parameter values yielding to a period function with opposite monotonicity at the outer boundary. This completes the proof of the result. ■

As we explain in the introduction, we conjecture that the parameters inside $\mathscr{R}_-$ and $\mathcal{T}_-$ are local regular values as well, i.e., that all the local bifurcation values lie on Ω_-.

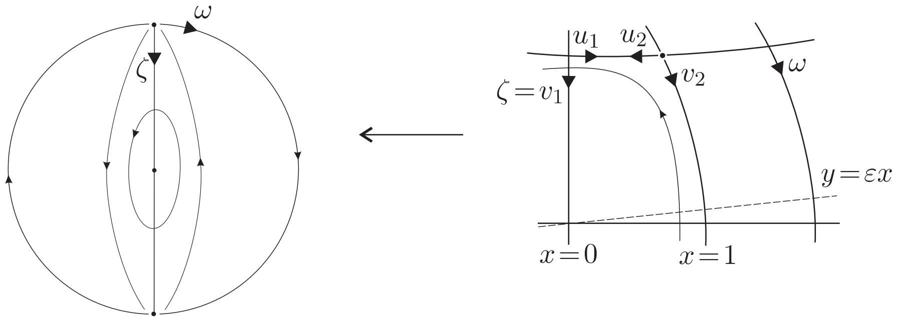

Figure 8: Desingularization and transverse sections in Lemma 3.11.

3.3 The case B > 0

In this case we set $B = 1$ without loss of generality. Note, see Figure 4, that the polycycle at the outer boundary of the period annulus contains finite hyperbolic saddles except for $\{F > 0, D > -1\}$. For these parameter values the polycycle consists of the straight lines $x = \pm 1$ together with their intersections at infinity, which are degenerated singular points. The next result treats this case and in its statement $T(s;\mu)$ is the period of the periodic orbit of system (1) passing through the point $(1-s, 0) \in \mathbb{R}^2$. Moreover we use the notation

$$\mathscr{R}_+ := \bigcup_{i \geqslant 0} \left\{ F = i\frac{n-1}{2} \right\}, \tag{10}$$

which is the set of resonant parameters in the normal form of the saddle that we obtain after desingularize the polycycle. The result follows by applying Theorem 3.6 and we point out that the case involving the Écalle-Roussarie compensator is missing because it would cover $F = n-1$, which is a resonant parameter

Lemma 3.11. *Define* $U_1 = \big\{D > -1, F > n-1\big\} \setminus \mathscr{R}_+$ *and* $U_2 = \big\{D > -1, F \in (0, n-1)\big\} \setminus \mathscr{R}_+$.

(*a*) *If* $\mu \in U_1$ *then* $T(s;\mu) = \Delta_1(\mu) + s\big(\Pi_1(\mu) + \mathcal{I}(U_1)\big)$, *where* Δ_1 *and* Π_1 *are analytic on* U_1. *In addition,* $\Pi_1(\mu) = 0$ *with* $\mu \in U_1$ *if, and only, if* $D = \frac{1-F}{2F-n}$ *and* Π_1 *is strictly negative* (*respectively, positive*) *for* $-1 < D < \frac{1-F}{2F-n}$ (*respectively,* $D > \frac{1-F}{2F-n}$).

(*b*) *If* $\mu \in U_2$ *then* $T(s;\mu) = \Delta_1(\mu) + s^{-\frac{F}{n-1}}\big(\Pi_2(\mu) + \mathcal{I}(U_2)\big)$, *where* Δ_1 *and* Π_2 *are analytic on* U_2. *Moreover,* $\Pi_2(\mu) = 0$ *with* $\mu \in U_2$ *if, and only, if* $F = 1$ *and* Π_2 *is strictly negative* (*respectively, positive*) *for* $0 < F < 1$ (*respectively,* $1 < F < n-1$).

Proof. To study this polycycle we use the chart $(\zeta, \omega) = \big(\frac{1}{y}, \frac{x}{y}\big)$, as explained in Section 2, and we obtain the vector field $\mathscr{L}(\zeta, \omega)$. The singular point at the origin is degenerated, so we blow up with $\{\omega = u_1 v_1, \zeta = v_1\}$. This yields to a vector field that has two hyperbolic saddles located at $(u_1, v_1) = (\pm 1, 0)$, which correspond to the invariant straight lines $x = \pm 1$ arriving to infinity, see Figure 8. Taking advantage of the symmetries of system (1), to study $T(s;\mu)$ for $s \approx 0$ it suffices to consider the time function associated to the passage through one of these saddles. To this end we perform a further translation $\{u_2 = 1 - u_1, v_2 = v_1\}$ to bring the one at $(u_1, v_1) = (1, 0)$ to the origin. Omitting the subscript of the last coordinates for the sake of simplicity, we thus obtain a vector field of the form (4) with $\{p = 0, q = 1\}$ and

$$P(u,v) = \frac{1 - (1-u)^{n-1}}{u} \text{ and } Q(u,v) = (u-1)\big(v^2 + F(1-u)^{n-3} + D(1-u)^{n-1}v^2\big).$$

Note that $\lambda=-\frac{Q(0,0)}{P(0,0)}=\frac{F}{n-1}$. Taking a small parameter $\varepsilon>0$, let us consider the transverse sections to the polycycle given by $\Sigma_\sigma=\{y=\varepsilon x\}$ and $\Sigma_\tau=\{x=0\}$, parameterized respectively with

$$s\longmapsto\big(1-s,\varepsilon(1-s)\big)\ \text{ and } s\longmapsto(0,1/s).$$

On account of $\big\{u=1-x,v=\frac{1}{y}\big\}$, in (u,v)-coordinates they write as $\sigma(s)=\big(s,\frac{1}{\varepsilon(1-s)}\big)$ and $\tau(s)=(1,s)$, respectively. Setting $\hat\mu=(F,D,\varepsilon)$, let $\hat T(s;\hat\mu)$ be the time that spends the solution of X_μ passing through the point $\sigma(s)\in\Sigma_\sigma$ to reach Σ_τ.

By applying Lemma 2.1 we can assert that $\widehat H(u,v):=H(1-u,\frac{1}{v})=A(1-u)+C(1-u)\frac{1}{v^2}$ is a first integral for $X_\mu(u,v)$. In order to study it we first note that

$$\frac{d}{du}A(1-u)=-\big(1-u+D(1-u)^n\big)\big(1-(1-u)^{n-1}\big)^{-1-\frac{2F}{n-1}}.$$

Since $u\longmapsto\frac{1-(1-u)^{n-1}}{u}$ is an analytic function at $u=0$, this shows that $\frac{d}{du}A(1-u)=u^{-\alpha}f_0(u)$ with $\alpha=1+\frac{2F}{n-1}$ and f_0 analytic at $u=0$. If $f_0(u)=\sum_{i\geqslant0}a_iu^i$ is its Taylor's series at $u=0$, then

$$A(1-u)=\sum_{i\geqslant0}\frac{a_i}{i-\alpha+1}u^{i-\alpha+1}\ \text{ provided }\alpha\notin\mathbb{N}.$$

This condition is precisely $F\notin\mathscr{R}_+$ and consequently $A(1-u)=u^{1-\alpha}f_1(u)$ with f_1 analytic at $u=0$. On the other hand, $C(1-u)=\frac{1}{2}\big(1-(1-u)^{n-1}\big)^{\frac{-2F}{n-1}}=u^{\frac{-2F}{n-1}}f_2(u)$ with f_2 analytic at $u=0$. Hence

$$\widehat H(u,v)=\frac{f_1(u)v^2+f_2(u)}{u^{\frac{2F}{n-1}}v^2}$$

is a generalized Darboux first integral for $X_\mu(u,v)$ with $\mu\in\{F>0,D>-1\}\setminus\mathscr{R}_+$ and this, on account of Remark 3.2, proves that the family of meromorphic vector fields under consideration satisfies the FLP.

Some computations show that

$$L(u)=\left(\frac{F+(1+D)u^2}{F+(1+D)\varepsilon^{-2}}\right)^{\frac{1}{2\lambda}}\ \text{ and } M(u)=\left(\frac{(n-1)u}{1-(1-u)^{n-1}}\right)^{\lambda}.$$

By applying Theorem 3.6, if $\mu\in U_1$ then $\hat T(s;\hat\mu)=\hat\Delta_1(\hat\mu)+s\hat\Pi_1(\hat\mu)+s\mathcal{I}(U_1\times V)$, where V is a neighbourhood of $\varepsilon=0$. In addition one can show that

$$\hat\Pi_1(\hat\mu)=\frac{\varepsilon}{1+D+F\varepsilon^2}+\int_0^{1/\varepsilon}\frac{\big((n-2)F+(nD+1)u^2\big)\big(F+(1+D)u^2\big)^{\frac{1}{2\lambda}-2}}{(D+1+F\varepsilon)^{\frac{1}{2\lambda}}}\frac{du}{u^{\frac{1}{\lambda}}}.$$

By construction we have $T(s;\mu)=4\lim_{\varepsilon\to0}\hat T(s;\hat\mu)$. One can prove that $T(s;\mu)=\Delta_1(\mu)+s\Pi_1(\mu)+s\mathcal{I}(U_1)$, with $\Delta_1(\mu)=4\lim_{\varepsilon\to0}\hat\Delta_1(\hat\mu)$ and

$$\begin{aligned}\Pi_1(\mu)=4\lim_{\varepsilon\to0}\hat\Pi_1(\hat\mu)&=\frac{4}{(D+1)^{\frac{1}{2\lambda}}}\int_0^{\infty}\big((n-2)F+(nD+1)u^2\big)\big(F+(1+D)u^2\big)^{\frac{1}{2\lambda}-2}\frac{du}{u^{\frac{1}{\lambda}}}\\&=\big(F-1+D(2F-n)\big)\frac{\sqrt{\pi}(n-1)}{\big(F(D+1)\big)^{3/2}}\frac{\Gamma\big(\frac{1}{2}-\frac{1}{2\lambda}\big)}{\Gamma\big(2-\frac{1}{2\lambda}\big)},\end{aligned}$$

where in the last equality we use twice (a) in Lemma 3.7. This proves (a) because the assertions concerning the sign of Π_1 follow easily using the properties of the Gamma function and that $\lambda(\mu)>1$ for $\mu\in U_1$.

Finally, if $\mu \in U_2$ then $\hat{T}(s;\hat{\mu}) = \hat{\Delta}_1(\hat{\mu}) + s^{\lambda}\hat{\Pi}_2(\hat{\mu}) + s^{\lambda}\mathcal{I}(U_2 \times V)$, where V is a neighbourhood of $\varepsilon = 0$ and

$$\hat{\Pi}_2(\hat{\mu}) = \sqrt{\frac{F}{(D+1)+F\varepsilon^2}}\left\{-\frac{1}{F}+\frac{1}{n-1}\int_0^1\left(\left(\frac{(n-1)u}{1-(1-u)^{n-1}}\right)^{\lambda+1}-1\right)\frac{du}{u^{\lambda+1}}\right\}.$$

As before, $T(s;\mu) = 4\lim_{\varepsilon\to 0}\hat{T}(s;\hat{\mu})$ and $T(s;\mu) = \Delta_1(\mu) + s^{\lambda}\Pi_2(\mu) + s^{\lambda}\mathcal{I}(U_2)$ with

$$\begin{aligned}\Pi_2(\mu) = 4\lim_{\varepsilon\to 0}\hat{\Pi}_2(\hat{\mu}) &= \frac{\sqrt{F}}{\sqrt{D+1}}\left\{-\frac{4}{F}+\frac{4}{n-1}\int_0^1\left(\left(\frac{n-1}{1-(1-u)^{n-1}}\right)^{\lambda+1}-\frac{1}{u^{\lambda+1}}\right)du\right\}\\ &= (n-1)^{\lambda-1}\frac{4\sqrt{F}}{\sqrt{D+1}}\frac{\Gamma(-\lambda)\,\Gamma\big(\frac{1}{n-1}\big)}{\Gamma\big(\frac{1}{n-1}-\lambda\big)},\end{aligned}$$

where to obtain the last equality we use (c) in Lemma 3.7. On account of $\lambda(\mu) \in (0,1)$ for $\mu \in U_2$, this shows that $\Pi_2(\mu) = 0$ with $\mu \in U_2$ if, and only if, $\lambda = \frac{1}{n-1}$, i.e., $F = 1$. The assertions concerning the sign are straightforward and are left to the reader. This shows (b) and completes the proof of the result. ■

Next result gives the behaviour near the polycycle of the period function of system (1) with $B = 1$, i.e.,

$$\begin{cases} \dot{x} = -y + x^{n-1}y, \\ \dot{y} = x + Dx^n + Fx^{n-2}y^2. \end{cases} \tag{11}$$

The proof for those parameter values with an unbounded period annulus follows directly from Lemma 3.11. In case of bounded period annulus the singularities at the polycycle are (finite) hyperbolic saddles, see Figure 4, and the proof follows verbatim the approach we used to prove Theorem 3.10. In the statement Ω_+ stands for the union of $\big\{D = \frac{1-F}{2F-n}, F > n-1\big\}$, $\big\{D > -1, F = 1\big\}$ and $\big\{D = -1, F \in (1, n-1)\big\}$. The definition of $\mathscr{T}_+$ and $\mathscr{R}_+$ is given in (3) and (10), respectively.

Theorem 3.12. *Consider the period function of the center at the origin of the differential system* (11). *Then the set* $\mathbb{R}^2 \setminus (\Omega_+ \cup \mathscr{R}_+ \cup \mathscr{T}_+)$ *corresponds to local regular values at the outer boundary of the period annulus. For these parameters the period function is monotonous near the outer boundary and the corresponding character is shown in Figure 7. Finally* Ω_+ *are local bifurcation values at the outer boundary.*

We conjecture that all the local bifurcation values at the outer boundary are inside Ω_+, in other words that $\mathscr{R}_+$ and $\mathscr{T}_+$ consists of local regular values.

4 Global behaviour of the period function

In this section we focus on the behaviour of the period function in its whole interval of definition. To this end we will take advantage of some previous results about the period function of a potential system. Our first result shows that we can indeed bring (1) to a potential system. It follows by applying [10, Lemma 5] and taking Lemma 2.1 into account, where the functions A, C and κ are defined.

Lemma 4.1. *The coordinate transformation* $\big\{u = f(x), v = \sqrt{2C(x)}y\big\}$, *where* $f(x) := \displaystyle\int_0^x \frac{\kappa(s)}{\sqrt{2C(s)}}\,ds$, *brings system* (1) *to the potential differential system*

$$\begin{cases} \dot{u} = -v, \\ \dot{v} = \big(A\circ f^{-1}\big)'(u). \end{cases} \tag{12}$$

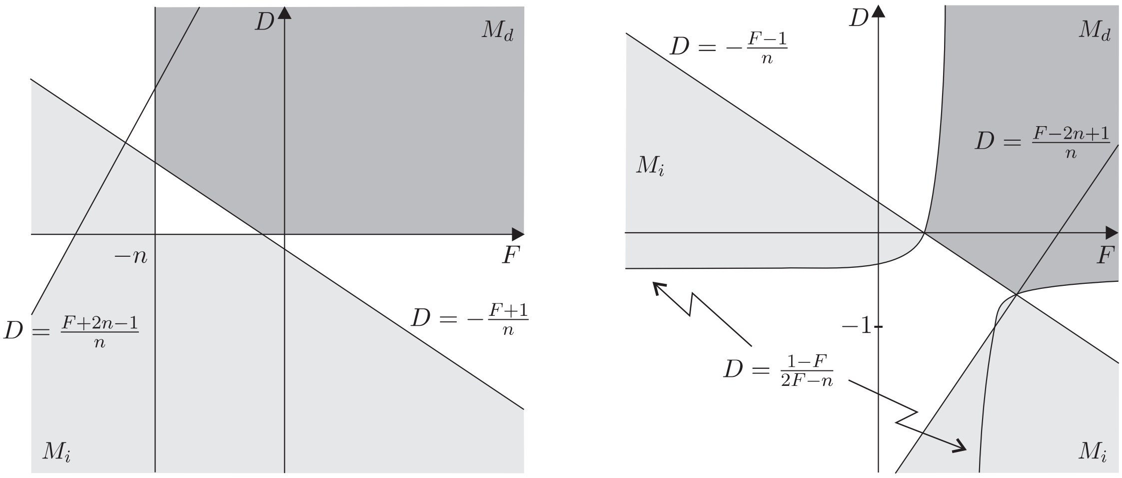

Figure 9: The subsets M_i and M_d for $B = -1$ (left) and $B = 1$ (right) in Proposition 4.3.

Lemma 4.2. *Let V be an analytic even function with $V(0) = 0$ and suppose that the differential system*

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = V'(x), \end{cases}$$

has a center at the origin. Let $(-x_r, x_r)$, with $x_r > 0$, be the projection of its period annulus on the x-axis. Then the following holds:

(a) *If the period function of the center is monotonous, then $x \longmapsto \frac{V(x)}{x^2}$ is monotonous on $(0, x_r)$.*

(b) *If $V^{(3)}$ does not vanish on $(0, x_r)$, then the period function of the center is monotonous.*

(c) *If $\mathcal{N} := 4\big(3(V'')^2 - V'V^{(3)}\big)V^2 - 12(V')^2V''V + 3(V')^4$ has at most one zero on $(0, x_r)$ counted with multiplicities, then the center has at most one critical period counted with multiplicities.*

Proof. The first assertion follows from a result by Opial, see [21, Theorem 6]. To show the other two we apply a criterion proved in [20]. To this end, set $\mu_1 := \frac{(V')^2 - 2VV''}{2(V')^2}$, and define $\mu_k = \frac{1}{2}\mu_{k-1} + \frac{1}{2k-3}\left(\frac{\mu_{k-1}V}{V'}\right)'$ for $k \geqslant 2$. Since V is an even function, by applying [20, Theorem A] it follows that if the number of zeros of μ_k on $(0, x_r)$, counted with multiplicities, is $n \geqslant 0$ and it is verified that $n < k$, then the number of critical periods of the center, counted with multiplicities as well, is at most n. The assertion in (b) is a particular case of the application of this criterion with $k = 1$, which shows that if μ_1 does not vanish on $(0, x_r)$, then the center does not have any critical period. To see this it suffices to note that the function $(V')^2 - 2VV''$ vanishes at $x = 0$ and that its derivative is equal to $-2VV^{(3)}$. This proves (b). Finally the assertion in (c) follows directly by applying the above criterion with $k = 2$ because one can verify that $\mu_2 = \frac{\mathcal{N}}{4(V')^4}$. This completes the proof of the result. ■

4.1 Monotonicity

We give now some regions of the parameter space where the period function is monotonous. As usual we reparametrize the case $B \neq 0$ to $B = \pm 1$. In the statement M_i (respectively, M_d) stands for the union of the light (respectively, dark) grey regions in Figure 9.

Theorem 4.3. *Consider the center at the origin of the differential system* (1)*. Then the following holds:*

(a) *If $B = \pm 1$ then the period function is monotonous increasing (respectively, decreasing) in case that (D, F) belongs to M_i (respectively, M_d).*

(b) *If $B = 0$ then the period function is monotonous increasing in case that $D > 0$ and $F > 0$, and it is monotonous decreasing in case that $(D, F) \in \{0 < nD < -F\} \cup \{nD < -|F|\}$.*

Proof. By Lemma 4.1, the coordinate transformation $\{u = f(x), v = \sqrt{2C(x)}y\}$ brings system (1) to

$$\begin{cases} \dot{u} = -v, \\ \dot{v} = V'(u), \end{cases} \tag{13}$$

with $V = A \circ f^{-1}$, where recall that A, C and κ are related to the first integral of (1) and are given in Lemma 2.1. Note in particular that V is an even function because n is odd. It is also clear that if $(-x_r, x_r)$ is the projection of the period annulus of system (1) on the x-axis, then the projection of the period annulus of (13) on the u-axis is $(-u_r, u_r)$ with $u_r = f(x_r)$.

We note at this point that $V'' = \eta \circ f^{-1}$ with $\eta := \frac{A''f' - A'f''}{(f')^3}$. Some long but straightforward computations show that

$$\eta(u) = 1 + (F + nD - B)u^{n-1} + D(F - nB)u^{2(n-1)}.$$

Our goal is to apply (b) in Lemma 4.2 taking advantage of the fact that $V^{(3)} = \left(\frac{\eta'}{f'}\right) \circ f^{-1}$. To this end, on account of $f' > 0$, note that $V^{(3)}$ is positive (respectively, negative) on $\big(0, f(x_r)\big)$ if, and only if, η is strictly increasing (respectively, decreasing) on $(0, x_r)$. One can check that

$$\eta'(x) = (n-1)x^{n-2}S(x^{n-1}) \text{ with } S(x) = F + nD - B + 2D(F - nB)x.$$

The value of x_r depends of course on $\mu = (B, D, F)$. For $B = \pm 1$ this value can be obtained by means of the study of the first portrait of (1) carried out in Section 2. Let us consider the case $B = -1$ first. Then, see Figure 4, we have that $x_r = +\infty$ for $D > 0$ and $x_r = (-D)^{\frac{-1}{n-1}}$ for $D < 0$. Accordingly, since S is linear, η' does not change sign on $(0, x_r)$ if, and only if,

$$D(F + nD + 1)(F + n) > 0, \text{ for } D > 0,$$

and $S(0)S(-1/D) > 0$, which writes as

$$(F + nD + 1)(nD - F - 2n + 1) > 0, \text{ for } D < 0.$$

For $B = 1$, see Figure 4, it turns out that $x_r = (-D)^{\frac{-1}{n-1}}$ for $D < -1$ and $x_r = 1$ for $D > -1$. Therefore η' does not change sign on $(0, x_r)$ if, and only if, $S(0)S(-1/D) > 0$, which writes as

$$(F + nD - 1)(nD - F + 2n - 1) > 0, \text{ for } D < -1,$$

and $S(0)S(1) > 0$, which writes as

$$(F + nD - 1)\big(F - 1 + D(2F - n)\big) > 0, \text{ for } D > -1.$$

Taking the sign of $S(0)$ also into account, Figure 9 displays the regions verifying these inequalities and this proves (a). Finally if $B = 0$, then it turns out that

$$V'\big(f(x)\big) = \left(\frac{A'}{f'}\right)(x) = (x + Dx^n)e^{\frac{F}{n-1}x^{n-1}}.$$

On account of this we have that $x_r = (-\frac{1}{D})^{\frac{1}{n-1}}$, for $D < 0$, and $x_r = +\infty$, for $D \geqslant 0$. Thus, η' does not change sign on $(0, x_r)$ if, and only if, $(F + nD)F > 0$ for $D > 0$ and $S(0)S(-1/D) > 0$, which writes as $(nD)^2 - F^2 > 0$, for $D < 0$. Accordingly, taking the sign of $S(0) = F + nD$ also into account, we obtain the inequalities in (b) and so the proof is completed. ■

4.2 Existence of critical periods

Our first result covers the case $B \neq 0$ and it is in fact a consequence of Theorems 3.10 and 3.12. The reader is referred to them for the definition of the curves $\Omega_\pm$, $\mathscr{R}_\pm$ and $\mathcal{T}_\pm$.

Theorem 4.4. *If (F, D) is inside the grey region in the left (respectively, right) diagram in Figure 7, but it does not belong to $\Omega_- \cup \mathscr{R}_- \cup \mathcal{T}_-$ (respectively, $\Omega_+ \cup \mathscr{R}_+ \cup \mathcal{T}_+$), then the period function of the differential system (1) with $B = -1$ (respectively, $B = 1$) has at least one critical period.*

Proof. The grey regions consists of parameters for which the period function has opposite monotonicity at the center and the polycycle. The monotonicity near the center is determined by the first period constant, which according to [24, Proposition 2.6] is given by $\Delta = B - F - nD$. The monotonicity near the polycycle follows from Theorems 3.10 and 3.12. ■

We conjecture that the monotonicity of the period function near the inner and outer boundary determines its global behaviour, i.e, that "nothing in between occurs". To be more precise, see Figure 7, our conjecture for $B = \pm 1$ is that there exists exactly one critical period in the grey regions and that the period function is monotonous in the complementary. The conjecture for $B = 0$ can be obtained by projecting the planes $B = \pm 1$ in Figure 7 on the unit sphere $B^2 + F^2 + D^2 = 1$ as explained in Section 1, and computing the points at the equator where the bifurcation curves land. For this reason we expect that if $B = 0$, then the period function of system (1) has exactly one critical period in case that $(F, D) \in \{-F < nD < 0\} \cup \{0 < -F < nD\}$ and it is monotonous otherwise. Concerning the first part of the conjecture for $B = 0$ we can prove the following:

Proposition 4.5. *If $B = 0$ then the period function of (1) has at least one critical period in case that $(F, D) \in \{-F < nD < 0\} \cup \{0 < -F < nD\}$.*

Proof. By applying Lemma 4.1, the change of variables $\{u = f(x), v = \sqrt{2C(x)}y\}$ brings system (1) to

$$\begin{cases} \dot{u} = -v, \\ \dot{v} = V'(u), \end{cases}$$

with $V := A \circ f^{-1}$, where recall that the functions A, C and κ are defined in Lemma 2.1. Note that V is an even function because n is odd. Let $(-u_r, u_r)$ with $u_r > 0$ be the projection of the period annulus of the center at the origin on the u-axis. By (a) in Lemma 4.2, if the period function is monotonous then $u \longmapsto \frac{A(f^{-1}(u))}{u^2}$ is monotonous on $(0, u_r)$. Clearly the latter occurs if, and only if, $x \longmapsto \frac{A(x)}{f(x)^2}$ is monotonous on $(0, x_r)$ with $x_r := f^{-1}(u_r)$. As we already showed in the proof of Proposition 4.3, we have that $x_r = +\infty$ for $D \geqslant 0$ and $x_r = (-D)^{\frac{-1}{n-1}}$ for $D < 0$. On the other hand,

$$\big(A'f - 2Af'\big)(x) = e^{\frac{F}{n-1}x^{n-1}} \left((x + Dx^n) e^{\frac{F}{n-1}x^{n-1}} \int_0^x e^{\frac{F}{n-1}s^{n-1}} ds - 2 \int_0^x (s + Ds^n) e^{\frac{2F}{n-1}s^{n-1}} ds \right).$$

Next we shall determine those parameters (F, D) for which the above function has opposite sign at $x = 0$ and $x \approx x_r$. This is a sufficient condition in order that $x \longmapsto \frac{A(x)}{f(x)^2}$ is non-monotonous on $(0, x_r)$. By computing the Taylor series of $A'f - 2Af$ we get that its sign near $x = 0$ is $F + nD$. If $D < 0$ then $x_r = (-\frac{1}{D})^{\frac{1}{n-1}}$, and therefore

$$\big(A'f - 2Af'\big)(x_r) = -2e^{\frac{-F}{(n-1)D}} \int_0^{(-\frac{1}{D})^{\frac{1}{n-1}}} (s + Ds^n) e^{\frac{2F}{n-1}s^{n-1}} ds < 0.$$

Consequently, if $F + nD > 0$ and $D < 0$, then the period function is non-monotonous. Finally, if $D \geqslant 0$ then $x_r = +\infty$ and

$$\lim_{x \to +\infty} e^{\frac{-F}{n-1}x^{n-1}} \big(A'f - 2Af'\big)(x) = -2 \int_0^{+\infty} (s + Ds^n) e^{\frac{2F}{n-1}s^{n-1}} ds < 0,$$

provided that $F<0$. Thus, we can assert that if $F+nD>0$ and $F<0$ then the period function is neither monotonous. This completes the proof of the result. ■

4.3 Uniqueness of critical period

The following result shows the validity of the conjecture posed in the introduction for parameters lying on $\{B=1, F=-\frac{n-1}{2}\}$ and $\{B=-1, F=\frac{n-1}{2}\}$.

Proposition 4.6. *Consider the center at the origin of system* (1)*. Then the following holds:*

(*a*) *If $B=1$ and $F=-\frac{n-1}{2}$, then the period function is monotonous for $D\leqslant\frac{n+1}{2n}$ and it has exactly one critical period for $D\in\big(\frac{n+1}{2n},n\big)$.*

(*b*) *If $B=-1$ and $F=\frac{n-1}{2}$, then the period function is monotonous for $D\notin\big(-\frac{n+1}{2n},0\big)$ and it has exactly one critical period for $D\in\big(-\frac{n+1}{2n},0\big)$.*

Proof. Once again we use that, by applying Lemma 4.1, $\big\{u=f(x), v=\sqrt{2C(x)}y\big\}$ brings system (1) to

$$\begin{cases} \dot{u}=-v,\\ \dot{v}=V'(u),\end{cases}\tag{14}$$

with $V=A\circ f^{-1}$, where the functions A, C and κ are given in Lemma 2.1. Recall also that if $(-x_r,x_r)$ is the projection of the period annulus of (1) on the x-axis, then the projection of the period annulus of (14) on the u-axis is $(-u_r,u_r)$ with $u_r=f(x_r)$. Our aim is to apply (*c*) in Lemma 4.2 and to this end the key point is that $F=-\frac{n-1}{2}B$ yields to $\kappa\equiv 1$, so that $A(x)=\frac{1}{2}x^2+\frac{D}{n+1}x^{n+1}$ and

$$f(x)=\int_0^x\frac{ds}{\sqrt{1-Bs^{n-1}}},$$

and this enables us to express the function $\mathcal{N}=4\big(3(V'')^2-V'V^{(3)}\big)V^2-12(V')^2V''V+3(V')^4$ in a very convenient way. Indeed, by computing implicitly the derivatives of $V=A\circ f^{-1}$, some long but easy computations show that $\mathcal{N}\big(f(x)\big)=(n-1)x^{n+5}p(x^{n+1})$, where $p(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5$ with

$$\begin{aligned}
a_0&=2(n+1)^2\big((n+1)B-2nD\big),\\
a_1&=4n(2n+1)(n-5)D^2+6(n+1)(n^2+6n+1)BD+(n-5)(n+1)^2B^2,\\
a_2&=4D\left(2n(n-7)D^2-(n+1)(5n^2-27n-2)BD+(n-6)(n+1)^2B^2\right),\\
a_3&=D^2\big((n+1)(15\,n^2-40\,n-47)B^2-(20\,n^2-168\,n-12)BD-(4\,n+12)D^2\big),\\
a_4&=8D^3\left((3n^2-11n-6)B+2(n+2)D\right),\\
a_5&=-32D^4B^2.
\end{aligned}$$

Let us consider the case $B=1$ first. We have in this case, see Figure 4, $x_r=(-D)^{\frac{-1}{n-1}}$ for $D<-1$ and $x_r=1$ for $D>-1$. Therefore, it suffices to study the zeros of p on the interval $I_+:=\big(0,s_r\big)$, where $s_r=-\frac{1}{D}$, for $D<-1$, and $s_r=1$, for $D>-1$. Let us note here that $s_r=\big(f^{-1}(u_r)\big)^{n+1}=x_r^{n+1}$. For each fixed n and D, let $\mathcal{Z}_+(n,D)$ be the number of zeros of p on I_+ counted with multiplicities. Note that $\mathcal{Z}_+$ can only change as we move n and D at those parameter values such that either $p(0)p(s_r)=0$, or p and p' have a common root, which is controlled by the discriminant $\mathscr{D}_+$ of p with respect to x. For $B=1$ we have that

$$p(0)=2(n+1)^2\big(n+1-2nD\big),\qquad p\left(\tfrac{-1}{D}\right)=-12(n-1)^3D\left(1+\tfrac{1}{D}\right)^2,$$

$$p(1)=3(n-1)(D+1)^2(n+1+2D)^2\qquad \mathscr{D}_+=D^{12}(D+1)^2(n+1+2D)^4\Delta_+(n,D),$$

with Δ_+ being a polynomial of total degree 16. Moreover it follows that, for each fixed $n \geqslant 9$, $\Delta_+(n, D) = 0$ has exactly two real roots $D = d_\pm(n)$ verifying that $\pm d_\pm(n) \in (n, n^3)$. (This can be shown by studying the zeros of the discriminant of Δ_+ with respect to D and the zeros of its leading coefficient in D. For the sake of shortness we do not include this study here.) The bifurcation curves $D = d_\pm(n)$, $D = -1$, $D = 0$, $D = -\frac{n+1}{2}$ and $D = \frac{n+1}{2n}$ split the half plane $\{n \geqslant 9, D \in \mathbb{R}\}$ in seven connected regions and, by construction, $\mathcal{Z}_+$ is constant in each one. Since one can verify that $\mathcal{Z}_+(10, -10^3)$, $\mathcal{Z}_+(10, -10)$, $\mathcal{Z}_+(10, -2)$, $\mathcal{Z}_+\big(10, -\frac{1}{2}\big)$ and $\mathcal{Z}_+\big(10, \frac{1}{2}\big)$ are zero, $\mathcal{Z}_+(10, 10) = 1$ and $\mathcal{Z}_+(10, 10^3) = 3$, we can assert that

$$\mathcal{Z}_+(n, D) = \begin{cases} 0 & \text{if } D \leqslant \frac{n+1}{2n}, \\ 1 & \text{if } D \in \big(\frac{n+1}{2n}, d_+(n)\big), \\ 3 & \text{if } D > d_+(n), \end{cases}$$

for each $n \geqslant 9$. On account of $d_+(n) > n$, by applying (c) in Lemma 4.2 this shows that the period function is monotonous for $D \leqslant \frac{n+1}{2n}$ and that it has at most one critical period for $D \in \big(\frac{n+1}{2n}, n\big)$. The fact that this critical period exists follows from Theorem 4.4 and this completes the proof of (a) for $n \geqslant 9$. The cases $n = 3$, $n = 5$ and $n = 7$ display more bifurcation values but one can arrive to the same conclusion.

Let us turn finally to the proof of (b). To this end note first that on account of (a) in Theorem 4.3 if $B = -1$, $D \notin \big(-\frac{n+1}{2n}, 0\big)$ and $F = \frac{n-1}{2}$, then the period function is monotonous. Accordingly it only remains to show that the period function has exactly one critical period for $D \in \big(-\frac{n+1}{2n}, 0\big)$. Since $D < 0$, Figure 4 shows that $x_r = (-D)^{\frac{-1}{n-1}}$. Thus by (c) in Lemma 4.2 it suffices to check that the number $\mathcal{Z}_-(n, D)$ of zeros of p on $\big(0, -\frac{1}{D}\big)$, counted with multiplicities, is 1. Exactly as before, $\mathcal{Z}_-$ can only change at those n and D such that $p(0)p\big(-\frac{1}{D}\big) = 0$ or the discriminant $\mathscr{D}_-$ of p with respect to x vanishes. Some computations show that $\mathscr{D}_-$ does not vanish for $D \in \big(-\frac{n+1}{2n}, 0\big)$. On the other hand, $p(0)p\big(-\frac{1}{D}\big) = 0$ if, and only if, $D \in \{-\frac{n+1}{2n}, 0, 1\}$. Consequently $\mathcal{Z}_-(n, D)$ does not change for $D \in \big(-\frac{n+1}{2n}, 0\big)$. Accordingly, since one can check that $\mathcal{Z}_-(10, -\frac{11}{40}) = 1$, by (c) in Lemma 4.2 there exists at most one critical period for $D \in \big(-\frac{n+1}{2n}, 0\big)$. We can assert that it exists thanks to Theorem 4.4 and this completes the proof of (b). ■

A The stereographic projection

Let $\langle\ ,\ \rangle$ stand for the scalar product in $\mathbb{R}^3$. Fix $p = (p_1, p_2, p_3) \in \mathbb{S}^2$ and denote the tangent plane to $\mathbb{S}^2$ at the point $-p$ by Π. For each $q \in \mathbb{S}^2 \setminus \{p\}$ we consider the intersection point $\bar{\sigma}_p(q)$ of the straight line $\ell := \{p + \lambda(q - p), \lambda \geqslant 0\}$ with the plane Π. Since $\langle \ell + p, p\rangle = 0$ yields to $\lambda = \frac{2}{1 - \langle p, q\rangle}$, we have that

$$\bar{\sigma}_p(q) = p + \frac{2(q - p)}{1 - \langle p, q\rangle} \in \Pi.$$

Assume that $\delta := \sqrt{p_1^2 + p_3^2} \neq 0$. Then $u_p := \frac{1}{\delta}(p_3, 0, -p_1)$ and $v_p := p \times u_p = \frac{1}{\delta}(-p_1p_2, p_1^2 + p_3^2, -p_2p_3)$ form an orthonormal basis of the plane Π. One can verify that $\bar{\sigma}_p(q) = -p + a_p(q)u_p + b_p(q)v_p$ with

$$a_p(q) = \langle u_p, \bar{\sigma}_p(q) + p\rangle = \frac{2}{\delta}\, \frac{p_1q_3 - p_3q_1}{p_1q_1 + p_2q_2 + p_3q_3 - 1}$$

and

$$b_p(q) = \langle v_p, \bar{\sigma}_p(q) + p\rangle = \frac{2}{\delta}\, \frac{p_1p_2q_1 - p_1^2q_2 - p_3^2q_2 + p_2p_3q_3}{p_1q_1 + p_2q_2 + p_3q_3 - 1},$$

where $q = (q_1, q_2, q_3)$. We thus obtain the expression of the *stereographic projection* $\sigma_p \colon \mathbb{S}^2 \setminus \{p\} \longrightarrow \mathbb{R}^2$, which is given by $\sigma_p(q) := \big(a_p(q), b_p(q)\big)$. As we explain in the introduction we choose $p = \hat{p}/\|\hat{p}\|$ with

$\hat{p} := (-1, -n, -2n)$ for the sake of convenience. Clearly the *radial projection* $\rho\colon \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{S}^2$ is given by

$$\rho(B, F, D) = \frac{1}{\sqrt{B^2 + F^2 + D^2}}(B, F, D).$$

Some computations show that $\tau := \sigma_p \circ \rho$ writes as $\tau(B, F, D) = 2\sqrt{\frac{1+5\,n^2}{1+n^2}}\Big(\tau_1(B, F, D), \tau_2(B, F, D)\Big)$ with

$$\tau_1(B, F, D) = \frac{nB - F}{\sqrt{1+5\,n^2}\sqrt{F^2 + D^2 + B^2} + nF + 2\,nD + B}$$

and

$$\tau_2(B, F, D) = \frac{(n^2+1)D - 2\,nB - 2\,n^2F}{\Big((1+5\,n^2)\sqrt{F^2 + D^2 + B^2} + \sqrt{1+5\,n^2}\,(nF + 2\,nD + B)\Big)}.$$

Note that τ maps $\mathbb{R}^3 \setminus \{\lambda\hat{p}, \lambda \geqslant 0\}$ to $\mathbb{R}^2$. Based on the curves in Figure 7, we take the restriction of τ on the planes $\{B = \pm 1\}$ to draw Figure 2. Similarly, but using the radial projection ρ as well, we draw Figure 1.

References

[1] M. Abramowitz and I. A. Stegun, "Handbook of mathematical functions with formulas, graphs, and mathematical tables", Dover, New York, 1992. (Reprint of the 1972 edition.)

[2] I. Boussaada, R. Chouikha and J. Strelcyn, *Isochronicity conditions for some planar polynomial systems,* Bull. Sci. Math. **135** (2011) 89–112.

[3] C. Chicone, *The monotonicity of the period function for planar Hamiltonian vector fields*, J. Differential Equations **69** (1987) 310–321.

[4] C. Chicone and M. Jacobs, *Bifurcation of critical periods for plane vector fields*, Trans. Amer. Math. Soc. **312** (1989) 433–486.

[5] C. Chicone, review in MathSciNet, ref. 94h:58072.

[6] C. Christopher and J. Devlin, *On the classification of Liénard systems with amplitude-independent periods*, J. Differential Equations **200** (2004) 1–17.

[7] A. Cima, F. Mañosas and J. Villadelprat, *Isochronicity for several classes of Hamiltonian systems*, J. Differential Equations **157** (1999) 373–413.

[8] A. Fonda, M. Sabatini and F. Zanolin, *Periodic solutions of perturbed Hamiltonian systems in the plane by the use of the Poincaré-Birkhoff Theorem*, Topol. Methods Nonlinear Anal. **40** (2012) 29–52.

[9] J.-P. Françoise and C. Pugh, *Keeping track of limit cycles*, J. Differential Equations **65** (1986) 139–157.

[10] A. Gasull, A. Guillamon and J. Villadelprat, *The period function for second-order quadratic ODEs is monotone,* Qual. Theory Dyn. Syst. **4** (2004) 329–352.

[11] A. Gasull, V. Mañosa and F. Mañosas, *Stability of certain planar unbounded polycycles*, J. Math. Anal. Appl. **269** (2002) 332–351.

[12] A. Gasull, C. Liu and J. Yang, *On the number of critical periods for planar polynomial systems of arbitrary degree*, J. Differential Equations **249** (2010) 684–692.

[13] M. Grau and J. Villadelprat, *Bifurcation of critical periods from Pleshkan's isochrones,* J. London Math. Soc. **81** (2010) 142–160.

[14] M. Grau, F. Mañosas and J. Villadelprat, *A Chebyshev criterion for Abelian integrals,* Trans. Amer. Math. Soc. **363** (2011) 109–129.

[15] J. Guckenheimer and P. Holmes, "Nonlinear oscillations, dynamical systems, and bifurcations of vector fields", Appl. Math. Sci. **42**, Springer-Verlag, New York, 1983.

[16] F. Mañosas and J. Villadelprat, *Bounding the number of zeros of certain Abelian integrals,* J. Differential Equations **251** (2011) 1656–1669.

[17] P. Mardešić, D. Marín and J. Villadelprat, *On the time function of the Dulac map for families of meromorphic vector fields,* Nonlinearity **16** (2003) 855–881.

[18] P. Mardešić, D. Marín and J. Villadelprat, *The period function of reversible quadratic centers,* J. Differential Equations **224** (2006) 120–171.

[19] P. Mardešić, D. Marín and J. Villadelprat, *Unfolding of resonant saddles and the Dulac time,* Discrete Contin. Dyn. Syst. **21** (2008) 1221–1244.

[20] D. Marín and J. Villadelprat, *On the return time function around monodromic polycycles,* J. Differential Equations **228** (2006) 226–258.

[21] Z. Opial, *Sur les périodes des solutions de l'équation différentialle* $x'' + g(x) = 0$, Ann. Polon. Math. **10** (1961) 49–72.

[22] R. Roussarie, "Bifurcation of planar vector fields and Hilbert's sixteenth problem", Progr. Math. **164**, Birkhäuser Verlag, Basel, 1998.

[23] J. Villadelprat, *On the reversible quadratic centers with monotonic period function*, Proc. Amer. Math. Soc. **135** (2007) 2555–2565.

[24] J. Villadelprat, *Bifurcation of local critical periods in the generalized Loud's system,* Appl. Math. Comput. **218** (2012) 6803–6813.

[25] C. Xingwu and W. Zhang, *Isochronicity of centers in a switching Bautin system*, J. Differential Equations **252** (2012) 2877–2899.

[26] Yan-Qian Ye *et al*, "Theory of limit cycles", Transl. Math. Monogr. **66**, American Mathematical Society, Providence, RI, 1986.

[27] Y. Zhao, *On the monotonicity of the period function of a quadratic system*, Discrete Contin. Dyn. Syst. **13** (2005) 795–810.